Directions: The allotted time is 80 minutes. Two sides of your own notes are allowed. No books or calculators are allowed, and please ask for help only if a question’s meaning is unclear.

1. (30 points) Consider the lowest two eigenstates of the infinite square well of width $a$,

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \text{ or } x > a \\ 0 & \text{if } 0 \leq x \leq a \end{cases} \quad (1)$$

whose wavefunctions are

$$\psi_1 = \sqrt{\frac{2}{a}} \sin(\pi x/a), \quad \psi_2 = \sqrt{\frac{2}{a}} \sin(2\pi x/a). \quad (2)$$

(a) Sketch the probability densities in space from these two wavefunctions and indicate the maxima, i.e., where the particle is most likely to be found in each.

(b) What is the expected value of the total energy in the state $\psi_2$? What is the variance of an energy measurement?

(c) Write the time evolution $\psi(x, t)$ of a system that starts at $t = 0$ in the state

$$\psi(x, 0) = \frac{1}{\sqrt{2}} (\psi_1(x) + i\psi_2(x)). \quad (3)$$

2. (40 points) Consider a particle in the infinite square well again. Let the square well potential be 0 from $-a/2$ to $a/2$ now, and let the particle start at $t = 0$ in the state

$$\psi(x) = A(a/2 - |x|). \quad (4)$$

(a) Normalize the wavefunction by finding $A$ (choose $A$ real and positive).

(b) Compute the inner products $\langle \psi_1 | \psi \rangle$, $\langle \psi_2 | \psi \rangle$, where

$$\langle f | g \rangle = \int_{-\infty}^{\infty} f^*(x)g(x) \, dx. \quad (5)$$

(c) What is the probability that a measurement of energy in this state gives the value $E_1$? (i.e., the energy of the lowest-energy eigenstate)

(d) What is the expectation value of the energy at $t = 0$? Make sure your answer is physically consistent.
(e) Is the expectation value of the energy constant in time in this state? Why or why not?

3. (a) (20 points) Find a scattering energy eigenstate \( \psi \) with given energy \( E > 0 \) of the potential \( V(x) = \alpha \delta(x) \), where \( \alpha > 0 \), that is even: \( \psi(x) = \psi(-x) \). Do not worry about normalization since this should not be a normalizable state.

(b) (10 points) Prove that any energy eigenstate that is nondegenerate and bound (normalizable) can be chosen to have a real wavefunction (i.e., so that the imaginary part is everywhere zero). To say that an energy eigenstate is nondegenerate means that any two normalized wavefunctions with that energy are related by an overall phase, \( \psi_1 = e^{i\theta} \psi_2 \), where \( \theta \) is constant.