No books, notes, or calculators are allowed. Please start each of the 4 problems on a fresh side. You should make an effort to answer every problem.

I. Consider a one-dimensional harmonic oscillator:

\[ H = \frac{p^2}{2m} + \frac{kx^2}{2}. \]  

(1)

Some useful facts: the energy levels of the one-dimensional harmonic oscillator are \( E = (n+1/2)\hbar\omega \), \( n = 0, 1, \ldots \), and the lowest two wavefunctions are of the form

\[ \psi_0 = \alpha_1 \exp(-\beta x^2), \psi_1 = \alpha_2 x \exp(-\beta x^2) \]  

(2)

where \( \alpha_1, \alpha_2, \beta \) are positive constants. All the following questions are for a single spinless particle.

(a) Using the exact values for the energy levels and \( \omega = \sqrt{k/m} \), find the change in the ground state energy induced by a small increase in the mass \( m \rightarrow m + dm \), to first order in \( dm \).

The ground state energy is \( \hbar \omega/2 = \hbar \sqrt{k/m}(1 - dm/2m) \), where \( \approx \) means that we have kept only linear order in \( dm \). So the change is \( \hbar \sqrt{k/m}(1 - dm/2m) \approx \hbar \sqrt{k/m} - dm/4m \).

(b) Explain how you would use first-order perturbation theory for the change in (a) by writing the change induced by \( dm \) as a perturbation. Write, but do not calculate, the integral that will give the first-order perturbation theory result for the energy change.

The perturbation Hamiltonian is \( -\frac{p^2}{2m}(dm/m) \). The first-order energy correction is then

\[ \langle \psi_0 | -\frac{p^2}{2m}(dm/m) | \psi_0 \rangle \]  

(3)

You can check that this is indeed equal to the above result, since the expected kinetic energy in the harmonic oscillator is half the total energy.

(c) Is the second order correction to the ground state energy positive, negative, or zero?

The second-order correction from the ground state is always either negative or zero. Since all even states are connected by the perturbation Hamiltonian in this case, the second-order shift is negative.

II. (a) Consider the 2p levels of the hydrogen atom and ignore fine structure corrections. Including spin, how many levels are there (for a single electron)?

There are six levels with \( \ell = 1 \) and \( s = 1/2 \), which we can label by two quantum numbers \( m_\ell = -1, 0, 1 \) and \( m_s = -1/2, 1/2 \). Without fine structure, these are all degenerate.

(b) Suppose that some interaction generates the perturbation

\[ H' = A \mathbf{L} \cdot \mathbf{S}. \]  

(4)

What are the energy shifts induced by this perturbation, to first order? What are the degeneracies?
Ignoring fine structure the six levels are initially degenerate. We can rewrite
\[ H' = AL \cdot S = A \frac{(L + S)^2 - L^2 - S^2}{2} = A(J^2 - L^2 - S^2)/2. \] (5)

Then the angular momentum addition rule tells us that the total momentum quantum number \( j \) takes the values \( j = 3/2 \) (4 states) or \( j = 1/2 \) (2 states). Since \( J^2 \) has the eigenvalue \( j(j+1)\hbar^2 \), and for all these states \( L^2 \) has eigenvalue \( \ell(\ell+1)\hbar^2 = 2\hbar^2 \) and \( S^2 \) has eigenvalue \( s(s+1)\hbar^2 = (3/4)\hbar^2 \), the \( j = 3/2 \) states are eigenstates of \( H' \) with eigenvalue \( Ah^2((3/2)(5/2) - 2 - 3/4)/2 = Ah^2/2 \), and the \( j = 1/2 \) states are eigenstates of \( H' \) with eigenvalue \( Ah^2((1/2)(3/2) - 2 - 3/4)/2 = -Ah^2/2 \). Working in the total angular momentum basis has diagonalized the perturbation Hamiltonian.

So the final energies are
\[ E = E_0 + Ah^2/2 \] (6)
for four levels, and
\[ E = E_0 - Ah^2 \] (7)
for two levels, where \( E_0 \) is the unperturbed energy (numerical value of \( E_0 \) for hydrogen \( n = 2 \) levels is \(-13.6 \text{ eV} / 4\)).

III. Consider a \( s = 1/2 \) fermion in a box: the particle is confined to move in one dimension between \( x = 0 \) and \( x = L \). The potential is zero in that region and infinite elsewhere. The one-particle energy eigenfunctions are
\[ \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \] (8)
and the energy levels are
\[ E_n = \frac{\pi^2 n^2 \hbar^2}{2mL^2}. \] (9)

(a) Consider the first-order energy shifts from a small sloping of the bottom of the box, i.e., a correction to the potential
\[ V'(x) = \alpha(x - L/2). \] (10)
Which (if any) energy levels will shift under this perturbation at first order? (Note that every level is still two-fold degenerate from spin.)

The first-order energy shift for orbital state \( n \) is
\[ \int_0^L |\psi_n(x)|^2 V'(x) \, dx. \] (11)
Since \( \psi_n(x) \) is either odd or even around \( x = a/2 \), \( |\psi_n(x)|^2 \) is even around \( x = a/2 \). The perturbation \( V'(x) \) is odd around \( x = a/2 \). Hence all these integrals are zero and there are no energy shifts. The two-fold spin degeneracy is unaffected by the perturbation, which does not involve spin.

(b) Write an expression for the energy shift of the (spin-degenerate) ground state to second order. Explain if any terms in the infinite sum must be zero by the symmetry of the original potential. You do not have to compute the nonzero terms in the energy shift.

The second-order perturbation theory result for the ground state energy is
\[ E_1^{(2)} = \sum_{j \neq 1} \frac{|H'_{1j}|^2}{E_1^{(0)} - E_j^{(0)}}. \] (12)
Again the spin degeneracy is unbroken. Now the ground state is even around \( x = a/2 \), so by the same argument as in (a), the matrix element of this perturbation with all other even states is zero. There are (in general) nonzero contributions from the odd states, which are labeled by even \( j \).

Plugging in the above formula for \( E_0^{(j)} \), we get

\[
E^{(2)}_1 = \sum_{\text{even } j} \frac{|H'_{j1}|^2}{\pi^2 h^2 (1 - j^2)},
\]

where

\[
H'_{j1} = \langle \psi_j^{(0)} | V' | \psi_j^{(0)} \rangle.
\]

(c) Now get rid of \( V' \) and suppose that the only perturbation is

\[
H' = gS_x.
\]

Possibly helpful: in the basis of \( S_z \) eigenstates, the \( S_x \) operator is represented by the matrix

\[
\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

What are the two lowest energy levels in the presence of this perturbation?

The orbital wavefunctions are now unmodified. The spin states that come from direct computation are the \( S_x \) eigenstates,

\[
| \uparrow_x \rangle = \frac{1}{\sqrt{2}} (| \uparrow \rangle + | \downarrow \rangle), \quad | \downarrow_x \rangle = \frac{1}{\sqrt{2}} (| \uparrow \rangle - | \downarrow \rangle).
\]

These have eigenvalues \( \pm \hbar/2 \), so the energy levels split from \( E_1 \) to \( E_1 \pm g\hbar/2 \). If the perturbation were increased in magnitude, the lowest energy levels would eventually be \( E_1 - g\hbar/2 \) and \( E_2 - g\hbar/2 \).

Since the question only asks for energy levels, it is perfectly reasonable to argue that rotational invariance of the unperturbed eigenstates implies that \( S_x \) should have the same eigenvalues as \( S_z \), which gives the above answer.

IV. Consider a particle in a two-dimensional harmonic oscillator with the same spring constant in the \( x \) and \( y \) directions:

\[
H = \frac{p_x^2 + p_y^2}{2m} + \frac{k}{2} (x^2 + y^2).
\]

Some useful facts: the energy levels of the one-dimensional harmonic oscillator are \( E = (n+1/2)\hbar\omega \), \( n = 0, 1, \ldots \), and the lowest two wavefunctions are of the form

\[
\psi_0 = \alpha_1 \exp(-\beta x^2), \quad \psi_1 = \alpha_2 x \exp(-\beta x^2)
\]

where \( \alpha_1, \alpha_2, \beta \) are positive constants. All the following questions are for a single spinless particle.

(a) What is the ground state energy of the 2D harmonic oscillator? What is the ground state wavefunction? Check that the wavefunction you write is an eigenstate.

The Hamiltonian is separable in \( x \) and \( y \) coordinates, i.e., of the form \( H_x + H_y \). We know the eigenstates \( \psi_n(x) \) and \( \psi_n(y) \) of these two parts, so a general eigenstate is

\[
\psi_m(x)\psi_n(y)
\]
with energy $\hbar \omega (m + 1/2 + n + 1/2) = \hbar \omega (m + n)$. The ground state energy is then $\hbar \omega$. The wavefunction is
\[ \psi_0(x)\psi_0(y), \] (21)
which is indeed found to be an eigenstate.

(b) What is the energy of the first excited state(s)? How many states have this energy? Again, write wavefunctions for the state(s).

There are two ways to make the next highest state: $m = 0, n = 1$ and $m = 1, n = 0$. These two states are degenerate and both have energy $2\hbar \omega$. Note that there is no symmetrization or antisymmetrization requirement for a single particle. The wavefunctions are
\[ \psi_1(x)\psi_0(y), \psi_0(x)\psi_1(y). \] (22)

(c) Compute the first-order perturbation theory change in energy levels, for the ground state and first excited state(s), under the perturbation
\[ H' = BL_z. \] (23)

It may help to recall that
\[ L = r \times p. \] (24)

For each energy level, identify the wavefunction corresponding to that energy.

The ground state is easy: it is nondegenerate and the expectation value of $L_z$ is zero, since the state is rotationally invariant:
\[ \psi_0(x)\psi_0(y) = \alpha_1^2 e^{-\beta(x^2+y^2)}. \] (25)

Here we have to compute the matrix of $L_z$ within the twofold degenerate space of first excited states. The two states are
\[ \alpha_1 \alpha_2 xe^{-\beta(x^2+y^2)}, \alpha_1 \alpha_2 ye^{-\beta(x^2+y^2)}. \] (26)

The explicit expression of $L_z$ is
\[ L_z = \frac{\hbar}{i} \left( \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \] (27)

So $L_z$ takes the exponential part, which is rotationally invariant, to zero, as in (a). We see an elegant result: $L_z$ takes the first wavefunction to $i\hbar$ times the second, and the second to $-i\hbar$ times the first. So the matrix of $L_z$ in these states is just
\[ \begin{pmatrix} 0 & i\hbar \\ -i\hbar & 0 \end{pmatrix}. \] (28)

Solving, the $L_z$ eigenvalues are $\pm \hbar$, the energy shifts are $B\hbar$ and $-B\hbar$, and the corresponding wavefunctions are
\[ \alpha_1 \alpha_2 (x + iy)e^{-\beta r^2}, \alpha_1 \alpha_2 (x - iy)e^{-\beta r^2}. \] (29)

If you prefer polar coordinates, note that the non-exponential part of the first is $re^{i\phi}$ and the second is $re^{-i\phi}$, consistent with $L_z = \pm \hbar$. 

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