We interpret \( |a(E)|^2 \) as the probability of finding the initial state with definite energy \( E \):

\[
|a(E)|^2 = \frac{\hbar}{2\pi} \frac{1}{(E_b - E)^2 + \frac{\hbar^2}{4\pi^2}}
\]

Normalize this to integrate to 1, the "Lorentzian" probability distribution is

\[
P(E) = \frac{\hbar}{2\pi\epsilon} \frac{1}{(E_b - E)^2 + \frac{\hbar^2}{4\pi^2}}
\]

Here \( \int_{-\infty}^{\infty} P(E) \, dE = 1 \). (The tail to \( E \rightarrow \pm\infty \) may be unphysical, but \( P(E) \) is peaked around \( E = E_b \) so this tail has little weight.)

This function looks like

\[
P(E) = \frac{1}{2} \frac{1}{P(E_b)}
\]

\[
\ln \left( \frac{(E_b - E)^2 + \frac{\hbar^2}{4\pi^2}}{2\frac{\hbar^2}{4\pi^2}} \right) = \frac{E - E_b}{\frac{\hbar}{2\pi}}
\]

\[
E = E_b \pm \frac{\hbar}{2\pi}.
\]
We can say a bit more about the Lorentzian distribution, which arises quite often in applications of QM. We derived it by expanding a \( \text{decaying state} \) of energy \( E_0 \) and lifetime \( \tau \) over energy eigenstates. We find a width \( \Delta E \propto \frac{\hbar}{\tau} \) (FWHM), which is an uncertainty in measurements of energy on the decaying state. Defining \( \Delta E \sim \frac{\hbar}{\tau} \), we see that \( (\Delta E)^2 \sim \frac{\hbar}{\tau} \),

which is a bit like the \( (\Delta x)(\Delta p) \sim \hbar \) uncertainty relation, although \( \tau \) is not the uncertainty of an observable.

Now we consider what mathematical properties of the Lorentzian distribution make it special. To motivate this physically, consider the measured photon spectrum for decays from one unstable state \( b \) to another unstable state \( a \).
When only the original state is unstable, with lifetime \( \tau \), it has an energy distribution

\[
P(E) = \frac{1}{2\pi \epsilon} \frac{1}{(E-E_0)^2 + \frac{1}{4}\epsilon^2}.
\]

By conservation of energy, the emitted photon has energy \( E - E_0 = \hbar \omega \), which is a Lorentzian peaked at \( E_0 - E_\omega \). What about transitions from a state with lifetime \( \tau_1 \) to lifetime \( \tau_2 \)? We can show that these the emitted photons have a Lorentzian distribution with \( \frac{1}{\epsilon^2} = \frac{1}{\tau_1} + \frac{1}{\tau_2} \).

To see this, first note that the "characteristic function" of a Lorentzian (the Fourier transform) is

\[
\varphi(k) = \int_{-\infty}^{\infty} \varphi(x) e^{ikx} dx.
\]

Let \( \varphi(x) = \frac{1}{\pi} \frac{\epsilon}{(x-x_0)^2 + \epsilon^2} \)

\[
= e^{ikx_0} \left( \frac{2\pi \epsilon i}{\pi} \left( \frac{1}{2\pi \epsilon i} e^{-\epsilon k^2} \right) \right) \text{ by contour integration}
\]

\[
= e^{ikx_0} e^{-\epsilon |k|}. 
\]
This Fourier transform is important because of the "Convolution Theorem":

the probability distribution $P(x)$ of the sum of two variables $x_1$ and $x_2$, with distributions $P_1(x_1)$ and $P_2(x_2)$, is

$$P(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_1(x_1)P_2(x_2) \delta(x_1 + x_2 - x) \, dx_1 \, dx_2$$

$$= \int_{-\infty}^{\infty} dx_2 \, P_1(x_1) P_2(x - x_1)$$

has the characteristic function $\phi(k) = \phi_1(k) \phi_2(k)$.

In our case, we want the probability distribution of the difference between $E_b$ and $E_a$
since $\Delta E = E_b - E_a$. Then we apply the convolution theorem to the two variables $E_b$ and $-E_a$:

$P_1(E_b)$ is $X_0 = E_b$, $a_1 = \frac{\hbar}{2 \tau_1}$

$P_2(E_a)$ is $X_0 = E_a$, $a_2 = \frac{\hbar}{2 \tau_2}$.

$$\phi_1(k) = e^{ikE_b} e^{-a_1 |k|}$$

$$\phi_2(k) = e^{-ikE_a} e^{-a_2 |k|}$$
\[ \phi(k) = e^{ik(E_b - E_a) - (q_1 + q_2)/2k} \]

which means \( P(E) \), the energy distribution of emitted photons, is a Lorentzian centered at \( E_b - E_a \), with

\[ \frac{1}{2\tau} = \frac{1}{2\tau_1} + \frac{1}{2\tau_2} \]

or

\[ \frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} \]
MANY- PARTICLE PHYSICS (Brensen ch. 10)

Now that we understand how single-electron atoms can interact with EM fields, we return to systems of multiple identical particles. Earlier we talked mostly about small numbers of particles; now our focus will be on statistically large numbers of particles.

Consider how a "gas" of fermions [at T=0 (zero temperature)] will be different from a "gas" of bosons. By "gas" we mean that the particles do not interact with each other.

Let's put the particles in a 3D cubic box of size L for definiteness: the energy levels are

$$\frac{\hbar^2}{2m} \pi^2 \left( n_x^2 + n_y^2 + n_z^2 \right). \quad n_x, n_y, n_z \geq 1.$$

For identical bosons, the ground state is to put all particles in the lowest single-electron state.

For spinless bosons, say, the state

$$|\Psi\rangle = \Phi_{n_1}(r_1) \Phi_{n_2}(r_2) \Phi_{n_3}(r_3) \ldots \Phi_{n_N}(r_N)$$
is a grand state, where \( \psi_{111} \) is the orbital wave function with \( n_x = n_y = n_z = 1 \).

What about fermions? Let's think about electron for definiteness. Since electron have \( s = \frac{1}{2} \), each orbital can have at most 2 electrons, one spin-up and one spin-down, from the Pauli principle. (The many-electron state is a Slater determinant made from the lowest single-particle states).

For \( N \gg 1 \) electrons, we want to understand which single-particle states \( n_x, n_y, n_z \) are occupied in the grand state. (This calculation is essentially the same as what we discussed before for Fermi's Golden Rule.) There will be some maximum energy \( E_F \) of single-particle state occupied:

\[
(n_x, n_y, n_z) \text{ occupied if } \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2) \leq E_F
\]

\[
\text{unoccupied if } \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2) > E_F.
\]

We want to calculate \( E_F(N) \), the "Fermi energy."
Now the idea of the calculation we did before was that the occupied \((n_x, n_y, n_z)\) grid points fill up \(\frac{1}{8}\) th of a sphere. Since \(\frac{\hbar^2 \pi^2 n^2}{2m L^2} = E_F\), where \(n\) is the radius of this sphere, \(n = \sqrt[3]{\frac{2m L^2 E_F}{\hbar^2 \pi^2}}\).

The volume of this sphere gives the number of grid points inside:

\[
\text{number of } (n_x, n_y, n_z) = \frac{1}{8} \left( \frac{4\pi n^3}{3} \right).
\]

Each orbital contains 2 electrons (this factor of 2 is the only difference from the calculation we did before), so the number of electrons is

\[
N = 2 \times \frac{1}{8} \times \left( \frac{4\pi n^3}{3} \right) = \frac{\pi n^3}{3} \quad = \frac{1}{3\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} V E_F^{3/2}, \quad V = L^3.
\]

Inverting this relationship, \(E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}\).
So the Fermi energy $E_F$ scales as the $\frac{2}{3}$ power of the particle density $\frac{N}{V}$. If we define the Fermi momentum $k_F$ so that $E_F = \frac{\hbar^2 k_F^2}{2m}$, we have

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}.$$  

This result means that if fermions are at sufficiently high density, the kinetic energy of the most energetic fermions may be very large simply because of the Pauli exclusion principle.

For example, electrons in a good metal like Al are well described as a Fermi gas, with $E_F \sim 10^4 - 10^5$ K much larger than room temperature. Fermi gases also appear in astrophysical situations. We can calculate the average energy of a Fermi gas as follows: the number of electrons in an interval $\Delta E$ is

$$\frac{dN}{dE} \times \Delta E = (\Delta E) D(E),$$

where $D(E)$ is what we had before, multiplied by 2 to take care of spin.
\[
\frac{dN}{dE} = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} VE^{1/2}
\]

The average energy of an electron in the ground state is

\[
\langle E \rangle = \frac{\int_0^{E_F} D(E)E \, dE}{N} = \frac{\text{total energy}}{\text{number of electrons}}.
\]

Writing \( N = \int_0^{E_F} D(E) \, dE \), we can cancel the constant factors and set

\[
\langle E \rangle = \sqrt[3]{\frac{\int_0^{E_F} E^{3/2} \, dE}{\int_0^{E_F} E^{1/2} \, dE}} = \frac{2}{3} \frac{E_{F}^{5/2}}{E_{F}^{3/2}} = \frac{2}{3} E_{F}
\]

\[
= \frac{3}{5} E_{F},
\]

so the mean energy is 60% of the maximum energy. The Fermi GS is a reasonable description of many problems as long as \( kT \ll E_F \),

thermal energy
Supposing, even the Bose grand state is a reasonable description of some system at small but nonzero temperature (cf. Physic 112 discussion of superfluid He 4).

For a physical picture, note that $k_F$ is of order $n^{1/3} = \frac{1}{\text{(interparticle spacing)}}$, so the $T=0$ energy scale $E_F$ for fermions is essentially just $E_F \approx \frac{\hbar^2}{2m} \left(\frac{1}{\text{(interparticle spacing)}}\right)^2$.

Of course, a Bose gas also has an interparticle spacing: is there any physical significance to the energy $E_B = \frac{\hbar^2}{2m} \left(\frac{1}{\text{(interparticle spacing)}}\right)^2$ for bosons?

It turns out that the answer is yes: if we convert $E_B$ to a temperature $T_B$ using $kT_B = E_B$, $k$ = Boltzmann's constant, then $T_B$ is a good estimate of the "Bose condensation temperature" below which many bosons are in the ground state.
A Fermi gas is a good description of the conduction electrons in a good metal like gold, since the conduction electrons move freely through the metal. The Fermi energy is often > 10,000 K, so at room temp. (300 K) the Pauli exclusion principle is more important than temperature in determining the electron energy.

Question: Why is $E_F$ so large? When we just said that $E_F = \frac{k^2}{2m} \frac{1}{(1/n)^2}$ ?

Answer: Because for electrons in a metal $m = m_e$, if we were talking about atoms ($m = \text{Amu}$), then $E_F$ would be much lower.

Now we turn to a fun application of the Fermi gas idea: STELLAR EQUILIBRIUM.

After a star has fused hydrogen into heavier elements that cannot be burned at stellar temperatures, it can either survive as a "dwarf" or else collapse into a "neutron star" or "black hole".
To understand when stars collapse, we need to assume a few facts: 1. the...energies as the kinetic energy of the electrons $\sim NE_F$ and the gravitational energy; 3. the relativistic ($\hbar k_F \gg mc$) generalization is $E_F = \hbar c k_F$ instead of $E_F = \frac{\hbar^2 k_F^2}{2m}$, but $k_F$ is unchanged = $(3\pi^2 n)^{1/3}$.

**Kinetic energy of electrons**

$$K.E. = NE_F = N\hbar c k_F = N\hbar c \left(\frac{3\pi^2 N}{3}\right)^{1/3}.$$  

$$V = \frac{4\pi R^3}{3}, \quad N = \frac{2M}{AM_p}, \quad Z = \text{atomic number}, \quad A = \text{mass number}, \quad M_p = \text{proton mass}.$$  

$$\Rightarrow K.E. = \frac{\hbar c \left(\frac{Z}{AM_p}\right)^{4/3} \left(\frac{9\pi}{4}\right)^{1/3}}{R} \frac{M^{4/3}}{R} = 6M^{4/3} R, \quad b = \hbar c \left(\frac{Z}{AM_p}\right)^{4/3} \left(\frac{9\pi}{4}\right)^{1/3}.$$  

To get the total gravitational energy, remember that the gravitational energy to focus $M_1, M_2 = -G M_1 M_2 / R$.  


and that the potential outside a sphere looks like the same as if the sphere were a point mass at the sphere center. Gravitational self-energy of a sphere: imagine building up the sphere by bringing in shells of mass until the sphere reaches its final radius \( R \). When the sphere is of radius \( r \), \( 0 \leq r \leq R \), then its mass is

\[
\frac{4\pi r^3 \rho}{3}
\]

\( \rho \) is mass density. Its potential with a spherical shell from \( r \) to \( r + dr \) is then

\[
\frac{G m_1 m_2}{r} = -\frac{G}{r} \left( \frac{4\pi r^3 \rho}{3} \right) \left( 4\pi (r^2 + dr^2) \rho \right)
\]

\[
= -\frac{16\pi^2 \rho^2 G}{3} r^4.
\]

Integrating this from \( r = 0 \) to \( r = R \) gives

\[
- \frac{16\pi^2 \rho^2 G}{3} \frac{R^5}{5} = -\frac{3}{5} \frac{G}{R} \left( \frac{4\pi R^3}{3} \right)^2
\]

\[
= -\frac{3}{5} \frac{6M^2}{R}.
\]

Combining this with the electron kinetic energy, we obtain

\[
E = \frac{6M^2}{R} - \frac{3}{5} \frac{6M^2}{R}.
\]
Since both tend to $\frac{1}{R}$, there are two possibilities
for $E(R)$:

I. $bM^{4/3} - \frac{3}{5} 6M^2 > 0$

Here $E \to \infty$ as $R \to 0$,
and there is no collapse.

II. $bM^{4/3} - \frac{3}{5} 6M^2 < 0$

and the star collapses.

So if $\frac{3 6M^2}{5} > bM^{4/3}$

$$\Rightarrow M > (\frac{5b}{36})^{3/2} = M_{\text{Chandrasekhar}}$$

the star collapses. If the star is slightly more
massive than $M_{\text{Chandrasekhar}}$, it collapses to a
very dense "neutron star" (the protons and electrons
combine to form extra neutrons): now the neutrons
form a Fermi gas. Very massive stars collapse to "black holes."
We now return to atoms and molecules to see how our methods apply to many-particle systems.

For a many-electron atom, the Pauli principle says that we should fill up orbitals of increasing energy as more electrons are added. For small atoms this seems to be correct; looking at the periodic table of elements,

- fill 1s orbital from H (Z=1) to He (Z=2)
- fill 2s from Li (Z=3) to Be (Z=4)
- fill 2p from B (Z=5) to Ne (Z=10)
- fill 2s, 3p from Na (Z=11) to Ar (Z=18)

But now we see a problem: potassium, element 19, has its last electron in 4s \(^{\text{\textsuperscript{1}}}\) rather than 3d.

This contradicts the single-electron energy formula:

\[ E = -\frac{13.6 \text{ eV}}{n^2} \]

Since \( E_{4s} > E_{3d} \).

How can we understand this? The actual order in which orbitals are filled in the periodic table depends on electron-electron interaction: it is

\[ 1s, 2s, 2p, 3s, 3p, [4s, 3d], 4p, [5s, 4d], 5p, [6s, 4f, 5d] \]

where orbitals in brackets \([ \cdot ]\) are filled concurrently.
There is a common approximation called the "central-field" approximation that gives some insight into why the orbitals fall in this way. The goal is to model the many-electron atom as a non-interacting electron problem in a modified potential \( V(r) \). The potential \( V(r) \) is chosen to be different from the nuclear Coulomb potential \(-\frac{2e^2}{r}\) in order to capture screening of the nucleus by inner electrons. The picture is the following: the outermost electrons spend most of the time at large distances from the nucleus, and at these distances the attraction of the nucleus is reduced by the inner electrons.

For the central-field approximation, we assume that potential screening is the dominant effect and can be summarized in the new central potential \( V(r) \). (For methods to obtain \( V(r) \), see Bransden 10.5). What changes if \( V(r) \) is not Coulombic?
When solving the Coulomb problem, we took advantage of spherical symmetry to break up \( Y(r, \theta, \phi) \) into \( Y(r, \theta, \phi) = R_n^l(r) Y_m(\theta, \phi) \).

For the Coulomb potential, the energy only depends on the quantum number \( n \), not on \( l \) or \( m \). Is this true in general if \( V(r) \) is not Coulombic?

No: for a general central potential, the energies depend on \( n \) and \( l \). There is no dependence on \( m \) because \( m \) does not appear in the radial equation that contains the energy eigenvalue: this \((2l+1)\)-fold degeneracy in the \( m \) quantum number reflects the invariance of energy under different directions of the angular momentum.

For a \( V(r) \) that describes a screened Coulomb potential, one finds that \( E(n, l) \) increases with \( l \). This explains why 2s fills before 2p, for example. In fact the dependence on \( l \) is strong enough that 3d and 4s have comparable energies.
We now outline the famous "Thomas-Fermi" method to obtain an approximate \( V(r) \) for a large atom, using our Fermi-yeo results for the electron density. This approach leads to a differential equation that has to be solved numerically, but the idea is quite simple.

Suppose that there are so many electrons that we can treat them statistically, using the Fermi-yeo result for a spatially variable electron density \( n(\mathbf{r}) \):

\[
E_F(\mathbf{r}) = \frac{\hbar^2}{2m} \left( 3 \pi^2 n(\mathbf{r}) \right)^{2/3}.
\]

The potential energy is \( e\Phi = -e\Phi \),

\[
\nabla^2 \Phi = 4\pi \rho(\mathbf{r}) = 4\pi \left( -en(\mathbf{r}) + Ze \delta(\mathbf{r}) \right).
\]

Setting \( E_F(\mathbf{r}) + (-e\Phi) = \text{const.} \) and taking \( \nabla^2 \) of both sides,

\[
\frac{\hbar^2}{2m} \nabla^2 \left( 3 \pi^2 n(\mathbf{r}) \right)^{2/3} + 4\pi Ze^2 n(\mathbf{r}) = 4\pi Ze^2 \delta(\mathbf{r}).
\]

This equation can be solved for the electron density numerically, and then substitution gives the potential \( \Phi(\mathbf{r}) \), which can be used as \( V(\mathbf{r}) \) in the central-field approximation.