Physics 137B, Fall 2007, Moore
Problem Set 11 Solutions

1.

For $\hat{\rho}_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ we have

$$\langle S_z \rangle = \text{Tr} \left[ \frac{\hbar}{2} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{\hbar}{2} \text{Tr} \left( \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix} \right) = 0,$$

$$\langle S_x \rangle = \text{Tr} \left[ \frac{\hbar}{2} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{\hbar}{2} \text{Tr} \left( \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \right) = \frac{\hbar}{2}.$$

For $\hat{\rho}_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ we have

$$\langle S_z \rangle = \text{Tr} \left[ \frac{\hbar}{2} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{\hbar}{2} \text{Tr} \left( \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right) = 0,$$

$$\langle S_x \rangle = \text{Tr} \left[ \frac{\hbar}{2} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{\hbar}{2} \text{Tr} \left( \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \right) = 0.$$

The fact that $\langle S_x \rangle = \hbar/2$ for $\hat{\rho}_1$ indicates that this density matrix represents a pure state with spin along the $+\hat{x}$ axis. Indeed, letting $|x, \uparrow\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$, we see that

$$|x, \uparrow\rangle \langle x, \uparrow| = \frac{1}{2} (|\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|) (|\uparrow\rangle + |\downarrow\rangle)$$

$$= \frac{1}{2} (|\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|) (|\uparrow\rangle + |\downarrow\rangle) (|\uparrow\rangle + |\downarrow\rangle) (|\downarrow\rangle \langle \downarrow| + |\uparrow\rangle \langle \uparrow|)$$
\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} = \hat{\rho}_1,
\]

(where \(|z, \uparrow\rangle\equiv |z, \uparrow\rangle\) are the usual eigenstates of \(S_z\)).

Now if we were to start in the state \(|x, \uparrow\rangle\) and measure \(S_z\), we know that the system will collapse either to \(|\uparrow\rangle\) or \(|\downarrow\rangle\), with probabilities

\[|\langle x, \uparrow | \uparrow \rangle|^2 = \frac{1}{2}\]

and

\[|\langle x, \uparrow | \downarrow \rangle|^2 = \frac{1}{2}.
\]

So if we measured \(S_z\) but didn’t look at the outcome of the measurement, we would say that our system is in a mixed state with 50% probability of being \(|\uparrow\rangle\) and 50% probability of being \(|\downarrow\rangle\). Alternatively, we could imagine creating a large number of identical copies of our system, all in the initial state \(|x, \uparrow\rangle\). Then if we measured \(S_z\) on all of them, but didn’t look at the results of the measurements, we would know that half of the copies were in the state \(|\uparrow\rangle\) and half were in the state \(|\downarrow\rangle\). In either case, this is precisely the mixed state that the density matrix \(\hat{\rho}_2\) represents.

4.

Suppose we have a mixed state of made up of pure states \(|\alpha_1\rangle, \ldots, |\alpha_m\rangle\) with nonzero probabilities \(W_1, \ldots, W_m, m > 1\). We will assume these states are orthogonal, so \(\langle \alpha_i | \alpha_j \rangle = \delta_{ij}\). The density matrix representing this system is

\[
\hat{\rho} = \sum_{i=1}^{m} W_i |\alpha_i\rangle \langle \alpha_i |.
\]

Its square is

\[
\hat{\rho}^2 = \left( \sum_{i=1}^{m} W_i |\alpha_i\rangle \langle \alpha_i | \right) \left( \sum_{j=1}^{m} W_j |\alpha_j\rangle \langle \alpha_j | \right)
\]

\[= \sum_{i,j=1}^{m} W_i W_j |\alpha_i\rangle \langle \alpha_i | \alpha_j \rangle \langle \alpha_j | \]

\[= \sum_{i=1}^{m} W_i^2 |\alpha_i\rangle \langle \alpha_i |.
\]

Since each \(W_i\) is nonzero and their sum is 1, each \(W_i\) must be strictly less than 1. Therefore \(W_i^2 \neq W_i\), and hence we see that \(\hat{\rho}^2 \neq \hat{\rho}\).
(Courtesy of Roger Mong)

**General Method** In all four cases, the potential $V(r)$ is spherically symmetric; $V(r)$ is not a function of the angle. The first born approximation is dependent only on the the momentum transferred $q$: ¹

$$f_B^{(1)}(q) = -\frac{1}{4\pi} \int e^{-iq \cdot r} U(r) \, d^3r \quad \text{(Bransden 13.151)}$$  \hspace{1cm} (1)

$$= -\frac{1}{q} \int_0^\infty U(r) \sin(qr) \, r \, dr \quad \text{(Bransden 13.156)}$$ \hspace{1cm} (2)

where:

$$U(r) = \frac{2m}{\hbar^2} V(r) \quad \text{(Bransden 13.22)}$$

$$q = |k' - k| = 2k \sin \frac{\theta}{2} \quad \text{(3)}$$

The scattering amplitude can be computed by evaluating the integral, or by identifying fourier transform pairs.

The differential cross section is simply the square modulus of the scattering amplitude:

$$\frac{d\sigma}{d\Omega} = |f_B|^2$$

The total cross section is computed by integrating the differential cross section over the entire sphere, where $|k'| = |k| = k$.

$$\sigma_{tot}(k) = \int \frac{d\sigma}{d\Omega}(k) \sin \theta \, d\theta \, d\phi$$

$$= \frac{2\pi}{k^2} \int_0^{2k} |f_B^{(1)}(q)|^2 q \, dq \quad \text{(Bransden 13.157)}$$ \hspace{1cm} (4)

The latter expression is a result of substituting $d\theta$ for $dq$ integral, applying azimuthal symmetry, and restricting the Born series to the 1st order.

¹Technically it’s $hq$ that is the momentum transferred, but the idea should be clear.
(a) Exponential potential

\[ V(r) = V_0 \exp(-\alpha r) \]

Amplitude:

\[ f_B^{(1)}(q) = -\frac{1}{q} \frac{2mV_0}{\hbar^2} \int_0^\infty e^{-\alpha r} \sin(qr) r \, dr \]

\[ = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i} \int_0^\infty \frac{r e^{-\alpha r} (e^{iqr} - e^{-iqr})}{e^{-\alpha r}} \, dr \]

Since \( \int_0^\infty x e^{-\beta x} \, dx = \frac{1}{\beta^2} \),

\[ f_B^{(1)}(q) = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i} \left( \frac{1}{(\alpha - iq)^2} - \frac{1}{(\alpha + iq)^2} \right) \]

\[ = -\left( \frac{2mV_0}{\hbar^2} \right) \frac{(\alpha^2 + 2i\alpha q - q^2) - (\alpha^2 - 2i\alpha q - q^2)}{2i q(\alpha^2 + q^2)^2} \]

Combining the terms:

\[ f_B^{(1)}(q) = -\left( \frac{2mV_0}{\hbar^2} \right) \frac{2\alpha}{(\alpha^2 + q^2)^2} \]

\[ \frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{4\alpha^2}{(\alpha^2 + q^2)^4} \]

\[ \sigma_{\text{tot}}(k) = \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 4\alpha^2 \int_0^{2k} \frac{q \, dq}{(\alpha^2 + q^2)^4} \]

\[ = \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 2\alpha^2 \left[ \frac{1}{3(\alpha^2 + q^2)^3} \right]_0^{2k} \]

\[ = \frac{4\pi\alpha^2}{3k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ \frac{1}{\alpha^6} - \frac{1}{(\alpha^2 + (2k)^2)^3} \right] \]

As \( k \to \infty \),

\[ \sigma_{\text{tot}}(k) \to \frac{4\pi}{3\alpha^4} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E} \]
(b) Gaussian potential

\[ V(r) = V_0 \exp(-\alpha^2 r^2) \]

Amplitude:

\[
 f_B^{(1)}(q) = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i} \left[ \int_0^\infty e^{-\alpha^2 r^2 + iqr} r dr - \int_{-\infty}^0 e^{-\alpha^2 r^2 - iqr} r dr \right] 
\]

In the last term, the substitution \( r \to -r \) is made, now both terms have the same integrand and the Gaussian integral can be computed by completing the square.\(^2\)

\[
 f_B^{(1)}(q) = -\left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i} \int_{-\infty}^\infty r \exp(-\alpha^2 r^2 + iqr) dr 
\]

\(^2\)Alternately, we note that the integrand in Eq.9 is even under inversion \( r \to -r \), so \( f_B^{(1)}(q) = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2} \int_{-\infty}^\infty e^{-\alpha^2 r^2} \sin(qr) r dr \). This makes the bounds of Gaussian integral from \(-\infty\) to \(\infty\).
\[
\begin{align*}
&= - \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i\alpha} \int_{-\infty}^{\infty} r \exp\left(-\alpha^2(r - \frac{q}{2\alpha^2})^2\right) \exp\left(-\frac{q^2}{4\alpha^2}\right) dr \\
&= - \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{2i\alpha} \left( e^{-\frac{q^2}{4\alpha^2}} \right) \int_{-\infty}^{\infty} \left( r - \frac{q}{2\alpha^2} + i \frac{q}{2\alpha^2} \right) \exp\left(-\alpha^2(r - \frac{q}{2\alpha^2})^2\right) dr
\end{align*}
\]

\( \int_{-\infty}^{\infty} (r - i \frac{q}{2\alpha^2}) \exp\left(-\alpha^2(r - i \frac{q}{2\alpha^2})^2\right) dr \) integrates to zero, leaving a shifted Gaussian. Shifting the integration path in the complex plane gives

\[
\int_{-\infty}^{\infty} \exp\left(-\alpha^2 r^2\right) dr = \sqrt{\pi} \alpha.
\]

\[
\begin{align*}
f^{(1)}_B(q) &= - \left( \frac{2mV_0}{\hbar^2} \right) \frac{\sqrt{\pi}}{4\alpha^3} e^{-\frac{q^2}{4\alpha^2}} \\
\frac{d\sigma}{d\Omega} &= \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{16\alpha^6} e^{-\frac{q^2}{2\alpha^2}}
\end{align*}
\]

(10)

(11)

The total cross section is an exponential integral:

\[
\begin{align*}
\sigma_{\text{tot}}(k) &= \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi^2}{16\alpha^6} \int_0^{2k} e^{-\frac{q^2}{2\alpha^2}} q dq \\
&= \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{16\alpha^4} \int_0^{2k} e^{-\frac{q^2}{2\alpha^2}} 2q dq \\
&= \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{16\alpha^4} e^{-\frac{q^2}{2\alpha^2}} \bigg|_0^{2k} \\
&= \frac{\pi^2}{8k^2\alpha^4} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left( 1 - e^{-\frac{2k^2}{\alpha^2}} \right)
\end{align*}
\]

(12)

As \( k \to \infty \),

\[
\sigma_{\text{tot}}(k) \to \frac{\pi^2}{8\alpha^4} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E}
\]

(13)
(c) Square-well\(^3\) potential

\[ V(r) = V_0 H(a - r) \]

\((H(x)\) is the Heaviside step function\)

\[ f_{B}^{(1)}(q) = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \int_0^a \sin(qr) \, r \, dr \]

Doing integration by parts with \(du = \sin(qr) \, dr\) and \(v = r\):

\[ f_{B}^{(1)}(q) = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \left[ -\frac{\cos(qr)}{q} r \right]_0^a - \int_0^a -\frac{\cos(qr)}{q} \, dr \]

\[ = -\frac{1}{q} \left( \frac{2mV_0}{\hbar^2} \right) \left[ -\frac{\cos(qr)}{q} r + \frac{\sin(qr)}{q^2} \right]_0^a \]

Which yields:

\[ f_{B}^{(1)}(q) = \left( \frac{2mV_0}{\hbar^2} \right) \left[ \frac{a}{q^2} \cos(qa) - \frac{1}{q^3} \sin(qa) \right] \quad (14) \]

\(^3\)‘Square-well’ is a weird name, considering the shape of this potential is a sphere. The potential looks like a square when graphing potential against radial distance. Technically it can’t even be described as a square since the axis don’t have matching units, but the name has stuck somehow.
\[
\frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ \frac{a}{q^2} \cos(qa) - \frac{1}{q^3} \sin(qa) \right]^2
\]

(15)

\[
\sigma_{\text{tot}}(k) = 2\pi \left( \frac{2mV_0}{\hbar^2} \right)^2 \int_0^{2k} \left[ \frac{a}{q^2} \cos(qa) - \frac{1}{q^3} \sin(qa) \right]^2 q \, dq
\]

Substitute \( x = qa \), the expression can be integrated\(^4\):

\[
\left( \frac{2mV_0}{\hbar^2} \right)^{-2} \sigma_{\text{tot}}(k) = \frac{2\pi}{k^2} \int_0^{2ka} \left[ \frac{a^3}{x^2} \cos(x) - \frac{a^3}{x^3} \sin(x) \right]^2 x \, dx
\]

\[
= \frac{2\pi a^4}{k^2} \int_0^{2ka} \left[ \frac{\cos^2 x}{x^3} - \frac{2(\sin x)(\cos x)}{x^4} + \frac{\sin^2 x}{x^5} \right] \, dx
\]

\[
= \frac{2\pi a^4}{k^2} \left[ \frac{1}{4x^2} + \frac{(\sin x)(\cos x)}{2x^3} - \frac{\sin^2 x}{4x^4} \right]_0^{2ka}
\]

\[
= \frac{2\pi a^4}{4k^2} \left[ \frac{x^2 - 2x(\sin x)(\cos x) + \sin^2 x}{x^4} \right]_0^{2ka}
\]

Analyzing the expression in the bracket:

\[
\lim_{x \to 0} \frac{x^2 - 2x(\sin x)(\cos x) + \sin^2 x}{x^4} = 1
\]

\[
\lim_{x \to \infty} \frac{x^2 - 2x(\sin x)(\cos x) + \sin^2 x}{x^4} = 0
\]

\[
\sigma_{\text{tot}}(k) = \frac{\pi a^4}{2k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ 1 - \frac{(2ka)^2 - 4ka(\sin 2ka)(\cos 2ka) + \sin^2 2ka}{(2ka)^4} \right]
\]

(16)

As \( k \to \infty \),

\[
\sigma_{\text{tot}}(k) \to \frac{\pi a^4}{2} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E}
\]

(17)

Note that despite having terms that blows up as \( q = 0 \), the amplitude (Eq.14) have a well behaved limit as \( q \to 0 \), as the singularities in terms cancel each other.

\(^4\)One can arrive at the antiderivative by trial and error. One heuristic is to perform integration by parts on the individual terms one at a time, seeing that terms in the integrand begin to simplify.
The total cross section (Eq.16) is also well behaved at \( k \to 0 \). The taylor expansion is:

\[
f^{(1)}_B(q) \sim a^3 \left( \frac{2mV_0}{\hbar^2} \right) \left[ -\frac{1}{3} + \frac{(qa)^2}{30} - \frac{(qa)^4}{840} + \cdots \right]
\]

\[
\sigma_{\text{tot}}(k) \sim \frac{2\pi a^4}{4k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ \frac{2(2ka)^2}{9} \frac{(2ka)^4}{45} + \frac{2(2ka)^6}{1575} + \cdots \right]
\]

(d) 'Polarisation' potential

\[ V(r) = \frac{V_0}{(r^2 + a^2)^2} \]

From part a), we know that the fourier transform of \( \exp(-\alpha r) \) gives \( \frac{8\pi \alpha}{(\alpha^2 + q^2)^2} \).\(^5\) Noting that the two expressions are fourier transform pairs, the fourier transform of \( \frac{8\pi \alpha}{(\alpha^2 + r^2)^2} \) gives \( (2\pi)^3 \exp(-\alpha q) \).\(^6\)

\[
f^{(1)}_B(q) = -\frac{1}{4\pi} \left( \frac{2mV_0}{\hbar^2} \right) \int e^{-iq \cdot r} \frac{1}{(r^2 + a^2)^2} d^3r
\]

\(^5\)Specifically: \( \int e^{-iq \cdot r} \exp(-\alpha r) d^3r = \frac{8\pi \alpha}{(\alpha^2 + q^2)^2} \).

\(^6\)\( \int e^{-iq \cdot r} \frac{8\pi \alpha}{(\alpha^2 + r^2)^2} d^3r = (2\pi)^3 \exp(-\alpha q) \). The factor of \( (2\pi)^3 \) is part of the definition of a fourier transform - though it’s placement depends on the convention.
\[- \frac{1}{4\pi} \left( \frac{2mV_0}{\hbar^2} \right) \frac{(2\pi)^3}{8\pi a} e^{-qa}\]

The scattering amplitude\(^7\) and cross section are thus:

\[ f^{(1)}_B(q) = - \left( \frac{2mV_0}{\hbar^2} \right) \frac{\pi}{4a} e^{-qa} \]  
\[ d\sigma \over d\Omega = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi^2}{16a^2} e^{-2qa} \]

\[ \sigma_{\text{tot}}(k) = \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{16a^2} \int_0^{2k} e^{-2qa} q \, dq \]
\[ = \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{64a^4} \int_{q=0}^{q=2k} e^{-2qa} (2qa) \, d(2qa) \]
\[ = \frac{2\pi}{k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{64a^4} \left[ e^{-2qa} (1 + (2qa)) \right]_{q=0}^{q=2k} \]
\[ = \frac{\pi^2}{32k^2a^4} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ 1 - e^{-4ka} (1 + (4ka)) \right] \]  \hspace{1cm} (21)

As \( k \to \infty \),

\[ \sigma_{\text{tot}}(k) \to \frac{\pi^2}{32a^4} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E} \]  \hspace{1cm} (22)

\(^7\)One can also evaluate the integral \( f^{(1)}_B(q) = - \left( \frac{2mV_0}{\hbar^2} \right) \frac{\pi}{4a} e^{-qa} \) using contour integration without using fourier transform properties. Doubling the integral to \((-\infty, \infty)\) and perform integration by parts\( du = \frac{r \, dr}{(r^2 + a^2)^2} \), \( v = \sin(qr) \) gives \( (2mV_0^2) \frac{1}{\hbar^2} \int_{-\infty}^{\infty} \cos(qr) \, dr \).

Closing the contour on the top/bottom in the complex plane: \( \left( \frac{2mV_0}{\hbar^2} \right) \int_{-\infty}^{\infty} \frac{\cos(qr) \, dr}{(r^2 + a^2)^2} \).

This method will give the same result with much more computation.
(Angular distribution)

The angular distribution is given by the differential cross section $\frac{d\sigma}{d\Omega}(q)$, where $q = 2k\sin\frac{\theta}{2}$. $q$ is constrained between $[0, 2k]$, with $q \to 0$ for forward scattering and $q \to 2k$ for backward scattering. Hence in all four cases and the Yukawa potential, the differential cross section is decreasing as the scattering angle increases\(^8\), which means that all the potentials (including Yukawa) favours\(^9\) forward scattering.

The differential cross section in each cases has a momentum scale for which it begins to drop off. This occurs roughly when $q \sim \alpha$ in the exponential and Gaussian potential, or when $q \sim a^{-1}$ in the square-well and the polarisation potential. $\alpha^{-1}$ or $a$ corresponds roughly to the 'size' of the scattering potential, beyond which the potential die off rapidly.

In the low energy limit, $k \ll \alpha$ or $a^{-1}$ and hence a small momentum transferred $q \to 0$. The differential cross section becomes roughly constant, with a weak dependence on the angle $\theta$. This implies that when the particle wavelength is much greater than the size of potential, the particle is scattered uniformly in all direction. In the high energy limit, $k \gg \alpha$ or $a^{-1}$, $\frac{d\sigma}{d\Omega}(2k) \to 0$. Only small momentum transfer $q$ will contribute to the cross section, strongly favouring a forward scattering process. This implies that when the particle wavelength pales in comparison to the potential size, the deflected particles will emerge in a small forward angle of about $\frac{\alpha}{k}$ or $\frac{1}{ka}$.

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\(^8\)Technically not true for the square-well, but the general idea is there.

\(^9\)Yes, Bransden uses British english. So does the solutions.
The Yukawa potential and the four potentials under scrutiny share many general aspect, but differ slightly in its angular distribution. Browsing over the graphs for each potential, the differential cross section may look similar to an untrained\(^{10}\) physicist, but the minute differences could mean finding the Higgs particle\(^{11}\).

(a) \(\frac{d\sigma}{d\Omega}\) for the exponential potential is a Lorentzian\(^{12}\) to the fourth power whereas the Yukawa potential only gives a Lorentzian squared. Hence the exponential exhibits a sharper cutoff in scattering angle than Yukawa.

(b) The \(\frac{d\sigma}{d\Omega}\) for the Gaussian and Yukawa potential are similar in the small \(q\) limit, but the it drops off exponentially as \(q \to \infty\) for the Gaussian whereas that of the Yukawa potential drops off as an inverse power \(q^{-4}\). This implies that for a high energy bombarding particles, a few particles will bounce backward in a Yukawa potential but practically none will in a Gaussian.

(c) The square-well differs from other potentials in that its \(\frac{d\sigma}{d\Omega}(q)\) is not a strictly decreasing function of \(q\). It drops to zero when \(qa\) is approximately an odd multiple\(^{13}\) of \(\pi\). This corresponds to certain angles which gives no scattering whatsoever. However, like the Yukawa potential, \(\frac{d\sigma}{d\Omega}\) still drops off as \(q^{-4}\) for large \(q\).

(d) In the small angle limit, \(\frac{d\sigma}{d\Omega}\) decreases linearly with the angle, as opposed to quadratically for the other cases. Also, \(\frac{d\sigma}{d\Omega}\) vanishes exponentially with large \(q\) (but not as fast as the Gaussian), compared to Yukawa’s negative fourth power.

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\(^{10}\)Saying that the exponential potential \(\frac{d\sigma}{d\Omega}\) drops off at \(\frac{q}{a} \sim 0.5\) compared to the Gaussian at \(\frac{q}{a} \sim 1\) is not an appropriate comparison, since \(\alpha\) (or \(a\)) in each potential has different meaning.

\(^{11}\)Obviously we won’t find the Higgs or SUSY looking at these graphs here. But this is essentially what particle experimentalist are working around the clock doing - analysing scattering data.

\(^{12}\)Also called Cauchy distribution. A Cauchy-Lorentz distribution has the form \(\frac{1}{2q^2 + 1}\).

\(^{13}\)More accurately, when \(qa = \tan(qa)\). (Or even more precisely, when \(q > 0\).)
(Summary)

This problem is so long, it requires a summary.

Yukawa:

\[
\frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{1}{(\alpha^2 + q^2)^2}
\]

(Bransden 13.162)

Exponential:

\[
f_B^{(1)}(q) = -\left( \frac{2mV_0}{\hbar^2} \right) \frac{2\alpha}{(\alpha^2 + q^2)^2}
\]

\[
\frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{4\alpha^2}{(\alpha^2 + q^2)^4}
\]

\[
\sigma_{\text{tot}}(k) = \frac{4\pi\alpha^2}{3k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ \frac{1}{\alpha^6} - \frac{1}{(\alpha^2 + (2k)^2)^3} \right]
\]

\[
\sim \frac{4\pi}{3\alpha^4} \left( \frac{2mV_0}{\hbar^2} \right) \frac{1}{E}
\]

Gaussian:

\[
f_B^{(1)}(q) = -\left( \frac{2mV_0}{\hbar^2} \right) \sqrt{\frac{\pi}{4\alpha^3}} e^{-\frac{q^2}{4\alpha^2}}
\]

\[
\frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi}{16\alpha^6} e^{-\frac{q^2}{2\alpha^2}}
\]

\[
\sigma_{\text{tot}}(k) = \frac{\pi^2}{8k^2\alpha^4} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left( 1 - e^{-\frac{2k^2}{\alpha^2}} \right)
\]

\[
\sim \frac{\pi^2}{8\alpha^4} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E}
\]

Square-well:

\[
f_B^{(1)}(q) = \left( \frac{2mV_0}{\hbar^2} \right) \left[ \frac{a}{q^2} \cos(qa) - \frac{1}{q^3} \sin(qa) \right]
\]

\[
\frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ \frac{a}{q^2} \cos(qa) - \frac{1}{q^3} \sin(qa) \right]^2
\]
\[ \sigma_{\text{tot}}(k) = \frac{\pi a^4}{2k^2} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ 1 - \frac{(2ka)^2 - 4ka(\sin 2ka)(\cos 2ka) + \sin^2 2ka}{(2ka)^4} \right] \]
\[ \sim \frac{\pi a^4}{2} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E} \]

Polarisation:

\[ f_B^{(1)}(q) = - \left( \frac{2mV_0}{\hbar^2} \right) \frac{\pi}{4a} e^{-qa} \]
\[ \frac{d\sigma}{d\Omega} = \left( \frac{2mV_0}{\hbar^2} \right)^2 \frac{\pi^2}{16a^2} e^{-2qa} \]
\[ \sigma_{\text{tot}}(k) = \frac{\pi^2}{32k^2a^4} \left( \frac{2mV_0}{\hbar^2} \right)^2 \left[ 1 - e^{-4ka} (1 + (4ka)) \right] \]
\[ \sim \frac{\pi^2}{32a^4} \left( \frac{2mV_0^2}{\hbar^2} \right) \frac{1}{E} \]