1.

(a) In the usual basis, the Hamiltonian is \( H = \frac{B\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). The initial state is \( |\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), so the time-evolved state is

\[
|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle = e^{-iBt/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

(b) Now the Hamiltonian is \( H = \frac{B\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), whose eigenstates are

\[
|\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},
\]

with eigenvalues \( E_{\pm} = \pm \frac{B\hbar}{2} \). Since we can expand the initial state as \( |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle) \), its time evolution is

\[
|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle = \frac{1}{\sqrt{2}}(e^{-iBt/2} |+\rangle + e^{iBt/2} |\rangle) = \begin{pmatrix} \cos \frac{Bt}{2} \\ -i \sin \frac{Bt}{2} \end{pmatrix}.
\]
The expectation value of $S_z$ for this state is therefore

$$\langle S_z \rangle = \langle \psi(t) | S_z | \psi(t) \rangle = \left( \cos \frac{Bt}{2} + i \sin \frac{Bt}{2} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( \cos \frac{Bt}{2} + i \sin \frac{Bt}{2} \right)$$

$$= \frac{\hbar}{2} \cos \frac{Bt}{2} + \left( -\frac{\hbar}{2} \right) \sin \frac{Bt}{2} = \frac{\hbar}{2} \cos Bt.$$

Remark: Once again, don’t forget to complex-conjugate your coefficients when you go from a ket to a bra. Many of you forgot to do this, and ended up with $\langle S_z \rangle = \hbar/2$ instead of the answer above.

2.

The unperturbed energy eigenstates of the infinite square well potential are

$$\psi^{(0)}_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \text{ with energies } E^{(0)}_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}.$$

The first-order energy shifts for the perturbation $H' = A\delta(x - L/2)$ are

$$E^{(1)}_n = \langle \psi^{(0)}_n | H' | \psi^{(0)}_n \rangle$$

$$= \int_0^L dx \frac{2}{L} \sin^2 \frac{n\pi x}{L} A\delta(x - L/2) = \frac{2A}{L} \sin^2 \frac{n\pi}{2}$$

$$= \begin{cases} 2A/L & n \text{ odd } \\ 0 & n \text{ even} \end{cases}.$$

By the same type of calculation we find that

$$| \langle \psi^{(0)}_n | H' | \psi^{(0)}_m \rangle |^2 = \left| \frac{2A}{L} \sin \frac{n\pi}{2} \sin \frac{m\pi}{2} \right|^2$$

$$= \begin{cases} (2A/L)^2 & n, m \text{ odd } \\ 0 & \text{otherwise} \end{cases}.$$
so the second-order energy shifts vanish for even \( n \), and for odd we have

\[
E^{(2)}_n = \sum_{m \neq n} \frac{|\langle \psi^{(0)}_n | H' | \psi^{(0)}_m \rangle|^2}{E^{(0)}_n - E^{(0)}_m} = \sum_{m \neq n \atop m \text{ odd}} \frac{(2A/L)^2}{2mL^2(n^2 - m^2)} = \frac{8mA^2}{\pi^2 h^2} \sum_{m \neq n \atop m \text{ odd}} \frac{1}{n^2 - m^2}.
\]

You can sum this last series explicitly. If you expand the summand in partial fractions and note that the infinite portions of the sums cancel, you’re left with the simple result

\[
\sum_{m \neq n \atop m \text{ odd}} \frac{1}{n^2 - m^2} = -\frac{3}{4n^2}.
\]

3.

First note that the given test functions \( \psi_\beta(x) = e^{-\beta^2x^2/4} \) are not normalized:

\[
\langle \psi_\beta | \psi_\beta \rangle = \int_{-\infty}^{\infty} dx \, e^{-\beta^2x^2/2} = \frac{\sqrt{2\pi}}{\beta}.
\]

The expectation value of the Hamiltonian in this state is

\[
E_\beta = \frac{\langle \psi_\beta | H | \psi_\beta \rangle}{\langle \psi_\beta | \psi_\beta \rangle}.
\]

To compute this, we’ll need to know how the Hamiltonian acts on the \( \psi_\beta(x) \):

\[
H \psi_\beta(x) = \left( \frac{\hbar^2}{2m} \frac{\beta^2}{2} \left( 1 - \frac{\beta^2x^2}{2} \right) + \frac{1}{2} kx^2 + \frac{1}{4} ux^4 \right) e^{-\beta^2x^2/4}.
\]

Using the integral identity provided, we find

\[
E_\beta = \frac{\beta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left[ \frac{\hbar^2}{2m} \frac{\beta^2}{2} \left( 1 - \frac{\beta^2x^2}{2} \right) + \frac{1}{2} kx^2 + \frac{1}{4} ux^4 \right] e^{-\beta^2x^2/2}
\]

\[
= \frac{\beta}{\sqrt{2\pi}} \left[ \frac{\hbar^2\beta^2}{4m} \cdot \frac{\sqrt{2\pi}}{\beta} + \left( \frac{1}{2} k - \frac{\hbar^2\beta^4}{8m} \right) \left( \frac{2}{\beta^2} \right)^{3/2} \cdot \frac{\sqrt{\pi}}{2} + \frac{1}{4} u \left( \frac{2}{\beta^2} \right)^{5/2} \cdot \frac{3\sqrt{\pi}}{4} \right]
\]

\[
= \frac{\hbar^2\beta^2}{8m} + \frac{k}{2\beta^2} + \frac{3u}{4\beta^4}.
\]
To get an estimate for the ground state energy, we need to minimize this expression with respect to $\beta$ (or, equivalently, with respect to $\beta^2$). Taking a derivative and setting the result equal to 0 gives the cubic equation

$$\frac{\hbar^2}{4m}(\beta^2)^3 - k(\beta^2) - 3u = 0. \tag{1}$$

This would be annoying to try to solve, so instead we assume that $u$ is a small parameter and expand the optimal choice of $\beta^2$ as a power series in $u$:

$$\beta^2 = a_0 + a_1 u + \ldots.$$ 

Inserting this expansion in (1), we find to first order in $u$ that

$$\left(\frac{\hbar^2}{4m}a_0^3 - ka_0\right) + \left(\frac{\hbar^2}{4m}3a_0^2a_1 - ka_1 - 3\right) u + O(u^2) = 0.$$ 

Equating each power of $u$ to zero yields

$$a_0 = \sqrt{\frac{4mk}{\hbar^2}} \quad \text{and} \quad a_1 = \frac{3}{2k}.$$ 

So to first order in $u$ the optimal value of $\beta$ is given by

$$\beta^2 = \frac{2}{\hbar}(mk)^{1/2} - \frac{3u}{2k} = \frac{2m\omega}{\hbar} - \frac{3u}{2m\omega^2},$$

(where we substitute $k = m\omega^2$), and the variational estimate of the ground state energy is

$$E_\beta = \frac{\hbar^2}{8m} \left(\frac{2m\omega}{\hbar} - \frac{3u}{2m\omega^2}\right) + \frac{m\omega^2}{2} \left(\frac{2m\omega}{\hbar} - \frac{3u}{2m\omega^2}\right)^{-1} + 3u \left(\frac{2m\omega}{\hbar} - \frac{3u}{2m\omega^2}\right)^{-2}.$$ 

Expanding the denominators to first order in $u$ we eventually obtain

$$E_\beta = \frac{1}{2}\hbar\omega + \frac{3h^2u}{16m^2\omega^2}.$$ 

This is exactly what you would have obtained from first-order perturbation theory.

Also note that you would get the same thing if you just substituted the zeroth-order solution $\beta^2 = \frac{2m\omega}{\hbar}$, rather than carrying the solution all the way to first order. I believe this is only a coincidence.
4.

(a) $|\ell - s| \leq j \leq \ell + s$, so $j = 3/2, 5/2, 7/2$ or $9/2$.

(b) There are 4 states with $m = 3/2$, one for each value of $j$.

(c) $L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2) = \frac{1}{2}(j(j + 1) - \ell(\ell + 1) - s(s + 1))\hbar^2 = \frac{\hbar^2}{2} \left( j(j + 1) - \frac{63}{4} \right)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$3/2$</th>
<th>$5/2$</th>
<th>$7/2$</th>
<th>$9/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L \cdot S$</td>
<td>$-6\hbar^2$</td>
<td>$-\frac{7}{2}\hbar^2$</td>
<td>$0$</td>
<td>$+\frac{9}{2}\hbar^2$</td>
</tr>
</tbody>
</table>

5.

The two lowest eigenfunctions for the one-particle harmonic oscillator are

$$
\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega^2x^2}{2\hbar} \right),
$$

$$
\psi_1(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \left( \frac{m\omega}{2\hbar} \right)^{1/2} x \exp \left( -\frac{m\omega^2x^2}{2\hbar} \right).
$$

(a) If the two spin-half particles are allowed to occupy different spin states, the lowest energy state is the antisymmetrized tensor product state

$$
|\phi\rangle = \frac{1}{\sqrt{2}} \left[ |\psi_0 \uparrow \rangle \otimes |\psi_0 \downarrow \rangle - |\psi_0 \downarrow \rangle \otimes |\psi_0 \uparrow \rangle \right],
$$

$$
\phi(x_1, x_2) = \frac{1}{\sqrt{2}} \psi_0(x_1)\psi_0(x_2)\left[ \chi_{\uparrow} \otimes \chi_{\downarrow} - \chi_{\downarrow} \otimes \chi_{\uparrow} \right].
$$

This state has energy $\frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \hbar\omega$. Plugging in the form of $\psi_0(x)$ given above you can easily check that it’s an eigenfunction of the Hamiltonian.
(b) If the two particles must both be spin up, the lowest energy state is

\[ |\phi\rangle = \frac{1}{\sqrt{2}} [|\psi_0 \uparrow\rangle \otimes |\psi_1 \uparrow\rangle - |\psi_1 \uparrow\rangle \otimes |\psi_0 \uparrow\rangle], \]

\[ \phi(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)] \chi_1 \otimes \chi_1. \]

This state has energy \( \frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega = 2\hbar\omega \). Since the wavefunction is now antisymmetric in \( x_1 \) and \( x_2 \), the wavefunction vanishes at \( x_1 = x_2 \). Under an interaction potential \( \Lambda \delta(x_1 - x_2) \), the first-order energy shift would vanish.