1.

We start with Eq. (8.10) in Bransden:

\[ H_0 \psi_n^{(1)} + H' \psi_n(0) = E_n^{(0)} \psi_n^{(1)} + E_n^{(1)} \psi_n^{(0)} \, . \]

In Dirac ket notation this reads

\[ H_0 |\psi_n^{(1)}\rangle + H' |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle \, . \]

If we hit this equation with the bra \( \langle \psi_n^{(1)} | \) we obtain

\[ \langle \psi_n^{(1)} | H_0 |\psi_n^{(1)}\rangle + \langle \psi_n^{(1)} | H' |\psi_n^{(0)}\rangle = E_n^{(0)} \langle \psi_n^{(1)} | \psi_n^{(1)} \rangle + E_n^{(1)} \langle \psi_n^{(1)} | \psi_n^{(0)} \rangle \, . \]

(Note that the first order corrections \( |\psi_n^{(1)}\rangle \) are not normalized, so we can’t simplify the first term on the right-hand side.) Rearranging, this reads

\[ \langle \psi_n^{(1)} | H' - E_n^{(1)} |\psi_n^{(0)}\rangle = -\langle \psi_n^{(1)} | H_0 - E_n^{(0)} |\psi_n^{(1)} \rangle \, . \]

Now taking the Hermitian conjugate, and using the fact that \( H_0 \) and \( H' \) are Hermitian, we have

\[ \langle \psi_n^{(0)} | H' - E_n^{(1)} |\psi_n^{(1)}\rangle = -\langle \psi_n^{(1)} | H_0 - E_n^{(0)} |\psi_n^{(1)} \rangle \, . \]

By Eq. (8.17), the left-hand side is precisely \( E_n^{(2)} \).
2.

For reference, we write the first two eigenfunctions for the one-dimensional harmonic oscillator, using Eq. (4.168) and \( \alpha = (m\omega/\hbar)^{1/2} \):

\[
\psi_0(x) = \left( \frac{\alpha}{\sqrt{\pi}} \right)^{1/2} e^{-\alpha^2 x^2/2}, \\
\psi_1(x) = \left( \frac{\alpha}{2\sqrt{\pi}} \right)^{1/2} 2\alpha x e^{-\alpha^2 x^2/2}.
\]

The eigenfunctions of the unperturbed two-dimensional harmonic oscillator

\[
H_0 = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} k (x^2 + y^2)
\]

are just products of the one-dimensional eigenfunctions, i.e.

\[
\Psi_{n_x n_y}(x, y) = \psi_{n_x}(x) \psi_{n_y}(y), \quad E_{n_x n_y} = \hbar\omega(n_x + n_y + 1).
\]

The ground state is clearly

\[
\Psi_{00}(x, y) = \psi_0(x) \psi_0(y) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2(x^2+y^2)/2},
\]

This ground state is non-degenerate, so the first order energy shift under the perturbation \( H' = \lambda xy \) is simply

\[
E_{00}^{(1)} = \langle \Psi_{00}|H'|\Psi_{00} \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \lambda xy |\Psi_{00}(x, y)|^2
\]

\[
= \frac{\lambda \alpha^2}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy xy e^{-\alpha^2(x^2+y^2)}
\]

\[
= \frac{\lambda \alpha^2}{\pi} \int_{-\infty}^{\infty} dx x e^{-\alpha^2 x^2} \int_{-\infty}^{\infty} dy y e^{-\alpha^2 y^2}.
\]

These integrals both vanish, since the integrand is odd, so the first order energy correction is \( E_{00}^{(1)} = 0 \).

The first excited level is doubly degenerate, spanned by the states \( |\Psi_{01}\rangle \) and \( |\Psi_{10}\rangle \) (both with energy \( 2\hbar\omega \)). To apply degenerate perturbation theory, we need to diagonalize the matrix

\[
\begin{pmatrix}
\langle \Psi_{01}|H'|\Psi_{01} \rangle & \langle \Psi_{01}|H'|\Psi_{10} \rangle \\
\langle \Psi_{10}|H'|\Psi_{01} \rangle & \langle \Psi_{10}|H'|\Psi_{10} \rangle
\end{pmatrix}.
\]
The first component is
\[
\langle \Psi_{01} | H' | \Psi_{01} \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \lambda xy |\psi_0(x)|^2 |\psi_1(y)|^2
\]
\[
= \frac{\lambda \alpha}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy xy (2\alpha y)^2 e^{-\alpha^2 x^2} e^{-\alpha^2 y^2}
\]
\[
= \frac{2\lambda \alpha^4}{\pi} \int_{-\infty}^{\infty} dx xe^{-\alpha^2 x^2} \int_{-\infty}^{\infty} dy y^3 e^{-\alpha^2 y^2}
\]
\[
= 0,
\]
since the \(x\) and \(y\) integrands are both odd. Similarly
\[
\langle \Psi_{10} | H' | \Psi_{10} \rangle = 2 \frac{\lambda \alpha^4}{\pi} \int_{-\infty}^{\infty} dx x^3 e^{-\alpha^2 x^2} \int_{-\infty}^{\infty} dy y^2 e^{-\alpha^2 y^2} = 0.
\]

The off-diagonal components are
\[
\langle \Psi_{01} | H' | \Psi_{10} \rangle = \frac{\alpha}{\sqrt{2}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \lambda xy (2\alpha x)(2\alpha y)e^{-\alpha^2 x^2} e^{-\alpha^2 y^2}
\]
\[
= \frac{2\lambda \alpha^4}{\pi} \int_{-\infty}^{\infty} dx x^2 e^{-\alpha^2 x^2} \int_{-\infty}^{\infty} dy y^2 e^{-\alpha^2 y^2}.
\]

To evaluate this integral, we use the hint from Problem set 3:
\[
\int_{-\infty}^{\infty} dx x^{2n} e^{-\beta^2 x^2/2} = \left( \frac{2}{\beta^2} \right)^{n+1/2} \Gamma(n + 1/2), \quad \Gamma(3/2) = \frac{\sqrt{\pi}}{2}.
\]

Then we find
\[
\langle \Psi_{01} | H' | \Psi_{10} \rangle = \frac{2\lambda \alpha^4}{\pi} \left( \frac{2}{2\alpha^2} \right)^{3/2} \frac{\sqrt{\pi}}{2} \left( \frac{2}{2\alpha^2} \right)^{3/2} \frac{\sqrt{\pi}}{2} = \frac{4\lambda}{\alpha^2}.
\]

Our degeneracy matrix now looks like
\[
\begin{pmatrix}
\langle \Psi_{01} | H' | \Psi_{01} \rangle & \langle \Psi_{01} | H' | \Psi_{10} \rangle \\
\langle \Psi_{10} | H' | \Psi_{01} \rangle & \langle \Psi_{10} | H' | \Psi_{10} \rangle
\end{pmatrix} = \begin{pmatrix} 0 & 4\lambda/\alpha^2 \\ 4\lambda/\alpha^2 & 0 \end{pmatrix},
\]
which has the obvious eigenvalues and eigenvectors
\[
E_+ = +4\lambda/\alpha^2, \quad |\phi_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]
\[
E_- = -4\lambda/\alpha^2, \quad |\phi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

In other words, the state \(\frac{1}{\sqrt{2}}(|\Psi_{01}\rangle + |\Psi_{10}\rangle)\) undergoes an energy shift of \(+4\lambda/\alpha^2\), whereas the state \(\frac{1}{\sqrt{2}}(|\Psi_{01}\rangle - |\Psi_{10}\rangle)\) undergoes a shift of \(-4\lambda/\alpha^2\).
3.

(a) The first \((n = 1)\) energy level of the hydrogen atom is 2-fold degenerate, spanned by the states
\[ |100 \uparrow \rangle, |100 \downarrow \rangle. \]
The second \((n = 2)\) energy level is 8-fold degenerate, with states
\[ |200 \uparrow \rangle, |200 \downarrow \rangle, |211 \uparrow \rangle, |211 \downarrow \rangle, |210 \uparrow \rangle, |210 \downarrow \rangle, |21, -1 \uparrow \rangle, |21, -1 \downarrow \rangle. \]

In a magnetic field, the hydrogen atom experiences a perturbing potential
\[ H' = -\frac{\mu_B B}{\hbar} (L_z + gS_z), \]
where \(\mu_B = e \hbar / 2m\) is the Bohr magneton and we take \(g = 2\). Since this perturbation commutes with the unperturbed Hamiltonian \(H_0\), we don’t have to worry about degenerate perturbation theory. The first-order energy corrections are just
\[ E_{nlm_{lms}}^{(1)} = \langle nlm_{lms} | H' | nlm_{lms} \rangle = -\mu_B B (m_l + 2m_s). \]

This gives the following table and the energy-level diagram in Figure 1.

| \(nlm_{lms}\) | \(|100 \uparrow \rangle\) | \(|100 \downarrow \rangle\) | \(|200 \uparrow \rangle\) | \(|200 \downarrow \rangle\) | \(|211 \uparrow \rangle\) |
|-------------|----------------|----------------|----------------|----------------|----------------|
| \(E_{nlm_{lms}}^{(1)}\) | \(-\mu_B B\) | \(+\mu_B B\) | \(-\mu_B B\) | \(+\mu_B B\) | \(-2\mu_B B\) |

| \(nlm_{lms}\) | \(|211 \downarrow \rangle\) | \(|210 \uparrow \rangle\) | \(|210 \downarrow \rangle\) | \(|21, -1 \uparrow \rangle\) | \(|21, -1 \downarrow \rangle\) |
|-------------|----------------|----------------|----------------|----------------|----------------|
| \(E_{nlm_{lms}}^{(1)}\) | \(0\) | \(-\mu_B B\) | \(+\mu_B B\) | \(0\) | \(+2\mu_B B\) |

(b) From the table the largest energy splitting in the \(n = 2\) level is between the \(|21, -1 \downarrow \rangle\) and \(|211 \uparrow \rangle\) states, with \(\Delta E = 4\mu_B B = 2e \hbar B / m\).

4.

(a) \(S_z\) has eigenvalues \(\pm \frac{\hbar}{2}\), so the Hamiltonian
\[ H_0 = g\frac{\mu_B B}{\hbar} S_z \]
Figure 1: Energy level diagram for hydrogen in a magnetic field.

has eigenvalues $\pm \frac{g \mu_B B}{2}$. The Bohr angular frequency is therefore

$$\omega = \frac{\Delta E}{\hbar} = g \mu_B B \hbar.$$ 

Plugging in $g = 2$, $B = 1$ Tesla, and $\mu_B = 9.27 \times 10^{-24}$ J/T (the Bohr magneton), we find

$$\omega \approx 1.76 \times 10^{-11} \text{ s}^{-1}.$$ 

(b) At time $t = 0$ we add a perturbation

$$H' = \frac{g \mu_B B'}{\hbar} S_x.$$
If the system is in the spin-down state at $t = 0$, then the first-order transition probability for finding the system in the spin-up state at time $t > 0$ is (Eq. (9.18) in Bransden)

$$P_{↑↓}(t) = \frac{\hbar}{2} \left| \int_0^t \langle ↑ \mid H' \mid ↓ \rangle \exp(i\omega t') dt' \right|^2$$

$$= \left( \frac{g\mu_B B'}{\hbar \omega} \right)^2 \sin^2 \frac{\omega t}{2}$$

$$= \left( \frac{B'}{B} \right)^2 \sin^2 \frac{\omega t}{2},$$

where we used the integral

$$\int_0^t e^{i\omega t'} dt' = \frac{1}{i\omega} (e^{i\omega t} - 1) = \frac{e^{i\omega t/2} - e^{-i\omega t/2}}{i\omega}$$

The maximum transition probability is just $(B'/B)^2 = 10^{-8}$. As a function of time, the transition probability oscillates between 0 and $10^{-8}$ with period

$$T = \pi/(\omega/2) = 2\pi/\omega.$$