The Arithmetical Hierarchy in the Setting of $\omega_1$ - Computability

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A.H. in $\omega_1$ - computability

- Joint work with Jacob Carson, Julia Knight, Karen Lange, Charles McCoy, John Wallbaum.

- *The Arithmetical hierarchy in the setting of $\omega_1$ - computability*, preprint.

- Continuation of work from N. Greenberg and J. F. Knight, *Computable structure theory in the setting of $\omega_1$*. 
Two definitions for the arithmetical hierarchy

We will give two definitions for the arithmetical hierarchy in the setting of $\omega_1$ - computability.

- The first will resemble the definition of the effective Borel Hierarchy.
- The second will resemble the standard definition of the hyper-arithmetical hierarchy.
Suppose $R$ is a relation of countable arity $\alpha$.

- $R$ is **computably enumerable** if the set of ordinal codes for sequences in $R$ is definable by a $\Sigma_1$ formula in $(L_{\omega_1}, \epsilon)$.

- $R$ is **computable** if it is both c.e. and co-c.e.
We assume that $P(\omega) \subseteq L_{\omega_1}$.

Results of Gödel give a computable 1-1 function $g$ from the countable ordinals onto $L_{\omega_1}$, such that the relation $g(\alpha) \in g(\beta)$ is computable.

So, computing in $\omega_1$ is essentially the same as computing in $L_{\omega_1}$. 
Indices for c.e. sets

- As in the standard setting, we have a c.e. set of codes for $\Sigma_1$ definitions.
- We write $W_\alpha$ for the c.e. set with index $\alpha$.
- All these definitions relativize in the natural way.
The jump

Definition

- We define the **halting set** as \( K = \{ \alpha : \alpha \in W_\alpha \} \).
- For a arbitrary set \( X \), \( X' = \{ \alpha : \alpha \in W^X_\alpha \} \).
- \( X^{(0)} = X \).
- \( X^{(\alpha+1)} = (X^{(\alpha)})' \).
- For limit \( \lambda \), \( X^{(\lambda)} \) is the set of codes for pairs \( (\beta, x) \) such that \( \beta < \lambda \) and \( x \in X^{(\beta)} \).

- We write \( \Delta^0_n \) for \( \varnothing^{n-1} \) for \( 1 \leq n < \omega \).
- We write \( \Delta^0_\alpha \) for \( \varnothing^\alpha \) for \( \alpha \geq \omega \).
Our first definition of the arithmetical hierarchy resembles the definition of the effective Borel hierarchy.

**Definition**

Let $R$ be a relation.

- $R$ is $\Sigma^0_0$ and $\Pi^0_0$ if it is computable.
- $R$ is $\Sigma^0_1$ if it is c.e.; $R$ is $\Pi^0_1$ if the complementary relation, $\neg R$, is c.e.
- For countable $\alpha > 1$, $R$ is $\Sigma^0_\alpha$ if it is a c.e. union of relations, each of which is $\Pi^0_\beta$ for some $\beta < \alpha$; $R$ is $\Pi^0_\alpha$ if $\neg R$ is $\Sigma^0_\alpha$. 
For \( \alpha \geq 1 \), we may assign indices for the \( \Sigma_\alpha^0 \) and \( \Pi_\alpha^0 \) sets in the natural way.

- For \( \alpha = 1 \), we write \((\Sigma, 1, \gamma)\) as the index for the c.e. set with index \( \gamma \).

- The set with index \((\Pi, 1, \gamma)\) is the complement.

- For \( \alpha > 1 \), the set with index \((\Sigma, \alpha, \gamma)\) is the union of sets with indices in \( W_\gamma \) of the form \((\Pi, \beta, \delta)\) for some \( \beta < \alpha \) and some countable \( \delta \).

- The set with index \((\Pi, \alpha, \gamma)\) is the complement.
Second definition for the arithmetical hierarchy

Our second definition for the arithmetical hierarchy resembles the standard definition for the hyper-arithmetic hierarchy.

Definition

Let $R$ be a relation.

- $R$ is $\Sigma^0_0$ and $\Pi^0_0$ if it is computable.
- $R$ is $\Sigma^0_1$ if it is c.e.; $R$ is $\Pi^0_1$ if $\neg R$, is c.e.
- For $\alpha > 1$, $R$ is $\Sigma^0_\alpha$ if it is c.e. relative to $\Delta^0_\alpha$; $R$ is $\Pi^0_\alpha$ if $\neg R$ is $\Sigma^0_\alpha$.

We assign indices for the $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ sets in the same way.
Comparing the two definitions

The two definitions agree at finite levels, but disagree at level $\omega$ and beyond.

- Under the first definition, membership of an element into a $\Sigma^0_\alpha$ set occurs if and only if that element is a member of one of the lower $\Pi^0_\beta$ sets.

- So membership into a $\Sigma^0_\alpha$ set uses information from a single lower level.

- Under the second definition, membership of an element into a $\Sigma^0_\alpha$ set may use a $\Delta^0_\alpha$ oracle to get information from all lower levels simultaneously.
The two definitions disagree at level $\omega$

**Proposition**

There is a set $S$ that is $\Delta^0_\omega$ under the second definition, but is not $\Sigma^0_\omega$ under the first definition.
Proof.

- Define $S$ such that $\alpha \in S$ iff $\alpha$ is not in the set with index $(\Sigma, \omega, \alpha)$ under the first definition.
- For each $n, \alpha$, let $S_{\alpha,n}$ be the union of the $\Sigma^0_n$ sets with indices in $W_\alpha$ of the form $(\Pi, k, \beta)$ with $k < n$.
- The union of these sets over all $n$ will be the set with index $(\Sigma, \omega, \alpha)$.
- A $\Delta^0_\omega$ oracle can determine whether $\alpha \in S_{n,\alpha}$ for all $n$. So $S$ is $\Delta^0_\omega$ under the second definition.
- However, $S$ cannot be one of the $\Sigma^0_\omega$ sets under the first definition.
The first definition of the computable infinitary formulas corresponds to the first definition of the arithmetical hierarchy.

**Definition**

Let $L$ be a predicate language with computable symbols. We consider $L$-formulas $\varphi(\bar{x})$ with a countable tuple of variables $\bar{x}$.

- $\varphi(\bar{x})$ is **computable** $\Sigma_0$ and **computable** $\Pi_0$ if it is a quantifier-free formula of $L_{\omega_1,\omega}$.
- For $\alpha > 0$, $\varphi(\bar{x})$ is **computable** $\Sigma_\alpha$ if $\varphi \equiv \bigvee (\exists \bar{u})\psi_i(\bar{u}, \bar{x})$, where each $\psi_i$ is computable $\Pi_\beta$ for some $\beta < \alpha$.
- $\varphi(\bar{x})$ is **computable** $\Pi_\alpha$ if $\varphi \equiv \bigwedge (\forall \bar{u})\psi_i(\bar{u}, \bar{x})$, where each $\psi_i$ is computable $\Sigma_\beta$ for some $\beta < \alpha$. 
Computable infinitary formulas

The second definition of the computable infinitary formulas corresponds to the second definition of the arithmetical hierarchy.

**Definition**

- $\varphi(x)$ is **computable** $\Sigma_0$ and **computable** $\Pi_0$ if it is a quantifier-free formula of $L_{\omega_1,\omega}$.
- For $\alpha > 0$, $\varphi(x)$ is **computable** $\Sigma_\alpha$ if $\varphi \equiv \bigvee_c (\exists u) \psi_i(u, x)$, where each $\psi_i$ is a **countable conjunction of formulas**, each computable $\Pi_\beta$ for some $\beta < \alpha$.
- $\varphi(x)$ is **computable** $\Pi_\alpha$ if $\varphi \equiv \bigwedge_c (\forall u) \psi_i(u, x)$, where each $\psi_i$ is a **countable disjunction of formulas**, each computable $\Sigma_\beta$ for some $\beta < \alpha$.
Using either one of the definitions for the computable infinitary formulas, the following proposition holds and is proved by induction on $\alpha$.

**Proposition**

Let $A$ be an $L$-structure, and let $\varphi(\vec{x})$ be a computable $\Sigma_\alpha$ (computable $\Pi_\alpha$) $L$-formula. Then the relation defined by $\varphi(\vec{x})$ in $A$ is $\Sigma^0_\alpha$ ($\Pi^0_\alpha$) relative to $A$. 
Relatively intrinsically arithmetical relations

**Definition**

- Let $\mathcal{A}$ be a computable structure, and let $R$ be a relation on $\mathcal{A}$.
- We say that $R$ is **relatively intrinsically** $\sum^0_\alpha$ on $\mathcal{A}$ if for all isomorphisms $F$ from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is $\sum^0_\alpha(\mathcal{B})$. 
We now present our main theorem.

**Theorem**

Let $1 \leq \alpha < \omega_1$. For a relation $R$ on a computable structure $A$, the following are equivalent:

1. $R$ is relatively intrinsically $\Sigma^0_\alpha$ on $A$.
2. $R$ is defined by a computable $\Sigma_\alpha$ formula.
Idea of the proof

- The theorem requires two proofs, one for each definition of the arithmetical hierarchy.
- In either case, the proof for $2 \Rightarrow 1$ follows directly from the proposition.
- This is because a computable $\Sigma_\alpha$ formula is $\Sigma^0_\alpha(B)$ for any structure $B$. So it must be relatively intrinsically $\Sigma^0_\alpha$ in $A$.
- The proof for $1 \Rightarrow 2$ invokes the use of forcing by building an isomorphism from a generic copy $B$ onto $A$, where our forcing elements are partial isomorphisms.
- The proof is similar to that of the analogous result in the standard setting.
Which definition is better?

- It is not very efficacious to have two definitions for the arithmetical hierarchy.
- The authors believe that the second definition is a more natural definition.
- Consider our previous construction of the set that highlighted the differences in the definitions.
- In the standard setting, an element enters a $\Sigma^0_\omega$ set based on finitely much $\Delta^0_\omega$ information.
- It seems natural that a membership into a $\Sigma^0_\omega$ set should use countably much $\Delta^0_\omega$ information.
References

- Greenberg, N. & Knight J. F., Computable structure theory in the setting of $\omega_1$, Proceedings of first EMU workshop, to appear.