The Law of Supply in Games, Markets and Matching Models*

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Summary. In large games with transferable utility, core payoffs satisfy a comparative statics property: If the proportion of one type of player increases, then the core payoff to that type of player decreases (does not increase). Markets with transferable utility satisfy a similar property: if the aggregate supply of a commodity increases, its value relative to the value of all commodities decreases. In market games, if one type of agent becomes more plentiful, his competitive payoff falls, and its decrease is engineered by a decrease in the relative value of his endowment.

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There is no result in general equilibrium theory that says that the price of a commodity will rise if its supply falls. That is, there is no "law of supply." Such a result is elusive for several reasons, including that all prices may change when the aggregate supply of one commodity falls, that the change in prices depends on which consumers lose endowments, and that the economy may permit multiple equilibria so that any comparison of prices requires a selection among equilibria.

However the idea that scarcity leads to high rents has been formalized in cooperative games with transferable utility. In this note we show how those results correspond to a law of supply in markets that generate such games, to market games (Shapley and Shubik [10] [11]) and to matching models. Before discussing markets we summarize a special case of the law of supply for games as derived by Engl and Scotchmer (ES, [3]).

Monotonicity and Comparative Statics in Games

Monotonicity means that when the relative numbers of players of different types change, the core payoffs to those types of players change in the opposite direction. The monotonicity result applies to aggregates. It is a result on the inner product of the difference in two population vectors and the difference in their core payoff vectors. 'Comparative statics,' which is not implied by monotonicity, means that if the proportion of one type of player increases, their core payoff falls.

A *player set* is a vector $\mathbf{N} = (N_1, ..., N_T) \in \mathbf{Z}_+^T \setminus \{0\}$ indicating the number of players of each of T types. A *coalition* is a vector $\mathbf{n} \in \mathbf{Z}_+^T \setminus \{0\}$. We assume that the total utility available to a coalition $\mathbf{n}$ is given by a superadditive function $V : \mathbf{Z}_+^T \setminus \{0\} \longrightarrow \mathbf{R}_+$. It will be useful to refer to the *composition* $s^n = \mathbf{n}/|\mathbf{n}|$ of a coalition $\mathbf{n}$ where $|\cdot|$ is the 1-norm i.e. $|\mathbf{n}| = \sum_t n_t$. A composition is in the simplex $\Delta = \{s \in \mathbf{R}_+^T \mid \sum_t s_t = 1\}$. We use int$\Delta$ to denote the interior of the simplex, $\{s \in \Delta \mid s_t > 0, \text{ for all } t\}$. It is convenient to define a function that represents the supremum of average payoffs as the size of the coalition varies, holding the composition $s \in \Delta$ fixed. Let $v : \Delta \longrightarrow \mathbf{R}_+^T$ by

$$v(s) = \sup\{V(rs)/r \mid r > 0, rs \in \mathbf{Z}_+^T\}.$$

We assume that v is not identically infinite everywhere on the interior of the simplex. By Lemma A.1 of ES the function v is concave, and by Lemma A.2, $V(rs)/r$ converges to v uniformly on compact sets in the interior of the simplex.

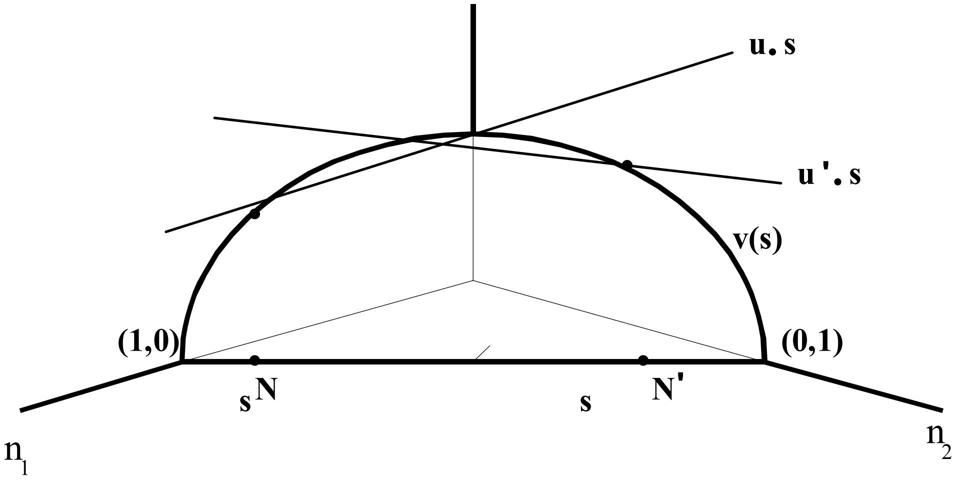

Figure 1:

A *game* is a pair, $(\mathbf{N}, V)$, where $\mathbf{N}$ is a player set and V is a function as above. The results that follow restrict attention to equal-treatment payoffs. An *equal-treatment payoff* (or simply *payoff*) is a vector $\mathbf{u} = (u_1, ..., u_T)$ in $\mathbf{R}^T_+$ such that $\mathbf{u} \cdot \mathbf{N} \leq V(\mathbf{N})$. Equal-treatment payoffs are of interest because competitive payoffs treat players of the same type equally, and because core payoffs also treat identical players almost equally in large games (ES, [3]).

A coalition $\mathbf{n}$ can *block* $\mathbf{u}$ if $\mathbf{u} \cdot \mathbf{n} < V(\mathbf{n})$. A payoff $\mathbf{u}$ is in the *core* of a game $(\mathbf{N}, V)$ if no coalition can block $\mathbf{u}$. We will let $C(\mathbf{N}, V)$ denote the core of a game $(\mathbf{N}, V)$, and compare the core payoffs in two games, say $(\mathbf{N}, V)$ and $(\mathbf{N}', V)$, where the player set $\mathbf{N}'$ has proportionately more players of particular type, say i, than the player set $\mathbf{N}$.

ES [3] show that in sufficiently large games, the core payoffs $\mathbf{u} \in C(\mathbf{N}, V)$ and $\mathbf{u}' \in C(\mathbf{N}', V)$ define linear functions that are "close" to supporting hyperplanes to the concave function v as shown in Figure 1. Their notion of "sufficiently large" is captured in the notion that the game "exhausts blocking opportunities." The set of blocking opportunities is $\Omega = \cup_{\varepsilon > 0} \Omega_\varepsilon$ where

$$\Omega_\varepsilon = \{\mathbf{u} \in \mathbf{R}^T_+ \mid \mathbf{u} \cdot s < v(s) - \varepsilon \text{ for some } s \in \Delta,\ d(s, \partial\Delta) \geq \varepsilon\}.^3$$

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[3] $d(s, \partial\Delta)$ is the distance from a point s to the boundary of Δ, $\partial\Delta = \Delta \setminus \text{int}\Delta$.

We say that a game $(\mathbf{N}, V)$ *ε-exhausts blocking opportunities* if every $\mathbf{u} \in \Omega_\varepsilon$ can be blocked. If the condition holds for $\varepsilon = 0$ (so every $\mathbf{u} \in \Omega$ can be blocked) then $\mathbf{u} \in C(\mathbf{N}, V)$ corresponds to a supporting hyperplane to v. With only ε-exhaustion, $\mathbf{u} \in C(\mathbf{N}, V)$ defines a linear function that is "close" to a supporting hyperplane. ES show that blocking opportunities are ε-exhausted in sufficiently large games.

The intuition for the following result is contained in Figure 1, which will be discussed below when we discuss Figure 2 on markets. We say that the player set $\mathbf{N}'$ has *proportionately more players of type i* than $\mathbf{N}$ if $s^{N'} = ks^N + kas_i^N \mathbf{e}^i$ for some $k, a > 0$ where $s^N = \mathbf{N}/|\mathbf{N}|$, $s^{N'} = \mathbf{N}'/|\mathbf{N}'|$ and e^i denotes the ith unit vector.

Proposition 1 (Comparative Statics in Games) *Suppose $T > 1$. Consider two games $(\mathbf{N}, V)$ and $(\mathbf{N}', V)$ such that $\mathbf{N}$ has proportionately more players of type i than $\mathbf{N}'$ and that $s^N, s^{N'} \in int\Delta$. Given $\gamma > 0$, there exists $\varepsilon > 0$ such that if $(\mathbf{N}, V)$ and $(\mathbf{N}', V)$ ε-exhaust blocking opportunities and $\mathbf{u} \in C(\mathbf{N}, V)$ and $\mathbf{u}' \in C(\mathbf{N}', V)$ then $u_i^{'} < u_i + \gamma$.*

The asymptotic monotonicity relationship $(s^N - s^{N'}) \cdot (\mathbf{u} - \mathbf{u}') \leq 2\gamma$ also holds. However this relationship does not imply comparative statics, and thus is not of particular interest.

The underlying assumptions on the game that are required for comparative statics and asymptotic monotonicity are minimal, namely superadditivity and bounded per-capita payoffs, and they include games derived from market games. Matching games and club economies may have more structure; namely they may satisfy a "scale" property that permits 0-exhaustion of blocking opportunities, and as a consequence permits a monotonicity conclusion on numbers of players as well as proportions. The scale property, which was introduced by Scotchmer and Wooders [9], implies that if a payoff can be blocked it can be blocked by a coalition smaller than a given maximum size. Wooders [13] has given the name "effective small groups" to this property. We shall say that the technology V has *effective small groups* if there exists a finite collection $\mathbf{C}$ of coalitions in $\mathbf{Z}_+^T \setminus \{0\}$, including the singleton coalitions, $e^1, \ldots, e^T$ such that the characteristic function $V : \mathbf{Z}_+^T \setminus \{0\} \longrightarrow \mathbf{R}_+$ satisfies

$$V(\mathbf{n}) = \max\{\sum_{j \in J} V(\mathbf{n}^j) \mid \sum_{j \in J} \mathbf{n}^j = \mathbf{n}, \mathbf{n}^j \in \mathbf{C}\}.$$

The following lemma and proposition restate the monotonicity theorem of Scotchmer and Wooders [9], except that they did not define the notion that a game exhausts blocking opportunities. The proofs are in our working paper [4].

Lemma 1 *Suppose a superadditive characteristic function has effective small groups. Then provided the player set* $\mathbf{N}$ *has the property that* $\mathbf{n}^j \leq \mathbf{N}$ *for every* $\mathbf{n}^j \in \mathbf{C}$*, the game* $(\mathbf{N}, V)$ 0-*exhausts blocking opportunities.*

Proposition 2 (Monotonicity in Games) *If* $(\mathbf{N}, V)$ *and* $(\mathbf{N}', V)$ 0-*exhaust blocking opportunities, and if* $\mathbf{u}$ *and* $\mathbf{u}'$ *are respectively in their cores, then* $(\mathbf{u} - \mathbf{u}') \boldsymbol{\cdot} (\mathbf{N} - \mathbf{N}') \leq 0$*.*

The Law of Supply in Markets

We shall take the law of supply to mean that the relative price of a commodity decreases when its supply increases, assuming that all other prices are also endogenous. This is in contrast to the "law of demand" which takes prices as exogenous rather than endogenous, and asks how aggregate demand changes as prices change, and makes no reference to whether markets clear. It is well known that the law of demand holds for quasilinear utility; however we show that the "law of supply" holds in a different sense.

We denote commodity bundles by vectors $x \in \mathbf{R}^\ell_+$. Suppose there is a finite set $\mathbf{M}$ of consumers and each consumer $i \in \mathbf{M}$ has a utility function $U^i : \mathbf{R}^\ell_+ \to \mathbf{R}$ and an endowment $\omega^i \in \mathbf{R}^\ell_+$. A *market* is a set of consumers, preferences and endowments, $(\mathbf{M}, (U^i)_{i \in M}, (\omega^i)_{i \in M})$. Given a price vector $p \in \mathbf{R}^\ell_+$ and an endowment $\omega \in \mathbf{R}^\ell_+$, we define a budget set $B(p, \omega) = \{x \in \mathbf{R}^\ell_+ \mid p \boldsymbol{\cdot} x \leq p \boldsymbol{\cdot} \omega\}$. Given a utility function U, a demand function is a function $F^U : \mathbf{R}^\ell_+ \times \mathbf{R}^\ell_+ \to \mathbf{R}^\ell_+$ such that $F^U(p, \omega) \in B(p, \omega)$ and $U(F^U(p, \omega)) \geq U(x)$ for any $x \in B(p, \omega)$. Thus F^{U^i} represents the demand of individual i. Aggregate demand is $\sum_{i \in \mathrm{M}} F^{U^i}(p, \omega^i)$.

When we say that there is a representative consumer we mean that aggregate demand can be derived from an aggregate utility function and, further, that the same aggregate utility function can be used whatever the initial endowments. This is a strong requirement, but the following lemma states that it is satisfied with quasilinear preferences. The preferences of agent i are *quasilinear* in the ℓ^{th} good if there exists $u^i : \mathbf{R}^{\ell-1}_+ \to \mathbf{R}$, such that preferences can be expressed as $U^i(x, q) = u^i(x) + q$ where $q \in \mathbf{R}$. Proofs of the following two lemmas can be found in our working paper [4]. The first is closely related to a result on the 'Gorman form' of utility cited by Varian [12].

Lemma 2 (Representative Consumer) *Suppose that all consumers' preferences* $\{U^i\}$ *are quasilinear and concave. Then there exists a quasilinear, concave* U *such that given any endowments* $\{\omega^i\}$*, aggregate demand is equal to* $F^U(p,\omega)$*, where* $\omega = \sum_i \omega^i$*. If the* U^i *are weakly monotone*[4] *or homogeneous of degree one, then so is* U*.*

For the case that there is a representative consumer, we define $\overline{\rho} : \mathbf{R}^\ell_+ \longrightarrow \mathbf{R}^\ell_+ \setminus \{0\}$ such that[5] $F^U(\overline{\rho}(\omega),\omega) = \omega$ and $U(\omega) = \overline{\rho}(\omega) \cdot \omega$. We also define $\rho(\omega) = \overline{\rho}(\omega)/|\overline{\rho}(\omega)|$. Thus, since demand is homogeneous of degree zero, $\rho(\omega)$ are market-clearing *relative* prices at supply ω. When U is quasilinear the prices $\overline{\rho}$ have the property that the quasilinear good is in fact the numeraire. (See our working paper for a proof of the following.)

Lemma 3 *Suppose there is a representative consumer with a utility function* U *that is quasilinear and homogeneous of degree 1. Then* $\overline{\rho}(z)_\ell = 1$ *for all* $z \in \mathbf{R}^T_+$*.*

As is well known, the following restricted version of the law of supply follows from revealed preference, provided U is weakly monotone. The remark is true because if $U(x) = U(y)$, then $\overline{\rho}(x) \cdot x \leq \overline{\rho}(x) \cdot y$ and $\overline{\rho}(y) \cdot y \leq \overline{\rho}(y) \cdot x$.

Remark 1 *If* $U(x) = U(y)$, $(\overline{\rho}(x) - \overline{\rho}(y)) \cdot (x - y) \leq 0$ *and* $(\rho(x) - \rho(y)) \cdot (x - y) \leq 0$.

The inequality in the Remark should be compared with the "law of demand," which asserts that $(p - p') \cdot (\sum_{i \in \mathrm{M}} F^{U^i}(p, \omega^i) - \sum_{i \in \mathrm{M}} F^{U^i}(p', \omega^i)) \leq 0$. In the law of demand prices are exogenous, whereas in the law of supply the aggregate supply is exogenous and prices are endogenous.

The law of supply expressed in Remark 1 is not very informative, as it does not tell us how prices change if aggregate supply changes in a way that does not hold utilities fixed. Further, it does not tell us that the price of a commodity falls if the commodity becomes proportionately scarcer. Nevertheless it exhibits the formal connection we will draw between prices and supplies. We let $s^x = x/|x|$ and $s^y = y/|y|$, and say that the aggregate supply vector x has *proportionately more of good* j than y if $s^x = ks^y + kas^y_j \mathbf{e}^j$ for $k, a > 0$.

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[4] By weakly monotone we mean $x \gg y \Longrightarrow U^i(x) > U^i(y)$ where $x \gg y$ means $x_i > y_i$ for all i.

[5] We take the existence of equilibrium here as primitive. Often convexity assumptions on preferences are made to insure the existence of equilibrium.

Proposition 3 (Monotonicity in Markets) *Suppose there is a representative consumer and that the aggregate utility function U is weakly monotone and homogeneous of degree 1. Let $x, y \in \mathbf{R}^{\ell}_{++}$. Then*

1. $\overline{\rho}(x) \boldsymbol{\cdot} y \geq \overline{\rho}(y) \boldsymbol{\cdot} y$ *and* $\overline{\rho}(y) \boldsymbol{\cdot} x \geq \overline{\rho}(x) \boldsymbol{\cdot} x$,
2. $(\overline{\rho}(x) - \overline{\rho}(y)) \boldsymbol{\cdot} (x - y) \leq 0$ *and*
3. $(\overline{\rho}(x) - \overline{\rho}(y)) \boldsymbol{\cdot} (s^x - s^y) \leq 0$.
4. *If x has proportionately more of good j than y, then $\overline{\rho}(x)_j \leq \overline{\rho}(y)_j$.*

Proof We first show that $\overline{\rho}(y)$ defines a bounding hyperplane for U at y; that is, $\overline{\rho}(y) \boldsymbol{\cdot} x \geq U(x)$ for all $x \in \mathbf{R}^{\ell}_{+}$ with equality at y. If $\overline{\rho}(y)$ were not a bounding hyperplane to U at y there would exist x such that $\overline{\rho}(y) \boldsymbol{\cdot} x < U(x)$ and therefore $\overline{\rho}(y) \boldsymbol{\cdot} rx < rU(x)$ for all $r > 0$. Let r^* satisfy $r^* U(x) = U(y)$. Since $\overline{\rho}(y) \boldsymbol{\cdot} r^* x < r^* U(x) = U(y) = \overline{\rho}(y) \boldsymbol{\cdot} y$, by monotonicity and homogeneity-of-degree-0 of demand functions, there exists $x' \gg r^* x$ such that $x' \in B(\overline{\rho}(y), y) = B(\rho(y), y)$ and $U(x') > U(r^* x) = r^* U(x) = U(y)$. This would contradict the definition of F^U. Second, since $\overline{\rho}(y)$ defines a bounding hyperplane at y and $\overline{\rho}(x)$, constructed similarly, defines a bounding hyperplane at x, $\overline{\rho}(y) \boldsymbol{\cdot} x \geq U(x) = \overline{\rho}(x) \boldsymbol{\cdot} x$ and $\overline{\rho}(x) \boldsymbol{\cdot} y \geq U(y) = \overline{\rho}(y) \boldsymbol{\cdot} y$, which demonstrates the first claim. The second and third claims follow from the inequalities in the first. To show the fourth inequality let $s^x = ks^y + kas^y_j e^j$ for $k, a > 0$ and use (1). In (1) subtract the second inequality from k times the first inequality to get $\overline{\rho}(y)_j \geq \overline{\rho}(x)_j$. $\square$

Figures 1 and 2 contain the intuition for the comparative statics in games and markets respectively (Propositions 1 and 3). Figure 2 shows a utility function whose domain is the $z_1 \times z_2$ plane. If the utility function is homogeneous, everything of relevance about it is summarized by its shape on the simplex, which is what we have drawn. Proposition 3(1) reports that $\overline{\rho}(x)$ corresponds to a supporting hyperplane to U at the aggregate supply vector x, hence (by homogeneity) at s^x and similarly for y. The monotonicity and comparative statics follow from how the gradient of the supporting hyperplane changes according to where it is tangent in the simplex Δ.

Proposition 3(2) and 3(3) follow because $\overline{\rho}(x)$ and $\overline{\rho}(y)$ are supporting hyperplanes. Figure 2 illustrates why Proposition 3(4) (comparative statics) holds. In Figure 2 the aggregate supply x has proportionately more of good 1 than the supply y, and proportionately

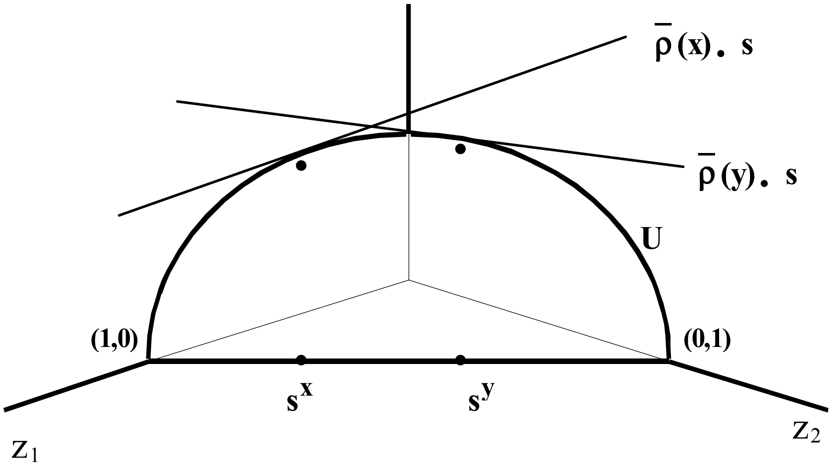

Figure 2:

less of good 2. We want to show that as a consequence $\overline{\rho}(x)_1 \leq \overline{\rho}(y)_1$ and $\overline{\rho}(x)_2 \geq \overline{\rho}(y)_2$. This follows from slopes of the linear functions that represent $\overline{\rho}(x)$ and $\overline{\rho}(y)$. The directional derivative of the linear function $z \mapsto \overline{\rho}(x) \bullet z$ in the direction from (1,0) to (0,1) is $(\overline{\rho}(x)_2 - \overline{\rho}(x)_1)/\sqrt{2}$, and similarly for the linear function $z \mapsto \overline{\rho}(y) \bullet z$. The slope of the line representing $\overline{\rho}(x)$ is larger than the slope of the line representing $\overline{\rho}(y)$, which implies that $\overline{\rho}(x)_2 - \overline{\rho}(x)_1 > \overline{\rho}(y)_2 - \overline{\rho}(y)_1$. We cannot have $\overline{\rho}(x) \gg \overline{\rho}(y)$, since that would imply that the line representing $\overline{\rho}(x)$ would lie everywhere above $\overline{\rho}(y)$, and then $\overline{\rho}(y)$ could not be a supporting hyperplane. Similarly we cannot have $\overline{\rho}(y) \gg \overline{\rho}(x)$. Therefore $\overline{\rho}(y)_1 \geq \overline{\rho}(x)_1$ and $\overline{\rho}(y)_2 \leq \overline{\rho}(x)_2$. The intuition for Proposition 1, comparative statics in games, is essentially the same as for markets.

There are two important reasons that the analogy between monotonicity of payoffs in games and monotonicity of prices in markets is imprecise, despite the similarity of Figures 1 and 2. First, the price-level for market-clearing prices is not determined, and whether the comparative statics result holds may depend on the normalization. The relationship $\overline{\rho}(x)_j \leq \overline{\rho}(y)_j$ does not necessarily imply that $\rho(x)_j \leq \rho(y)_j$ since it may be that $|\overline{\rho}(x)| < |\overline{\rho}(y)|$. Normalization is not an issue with core payoffs; $\mathbf{u}$ and $\mathbf{u}'$ will typically not sum to the same magnitude, but this is irrelevant since the magnitudes themselves (not just the relative magnitudes) are the objects of interest. Second, the choice of how to represent commodities is arbitrary, whereas the representation of a player as the integer 1 is natural. Whether the comparative statics hold may depend on the representation of commodities (see Proposition 4 below).

It is perhaps natural to think that since the law of demand holds for quasilinear preferences, the law of supply also holds. The law of supply is stated in the above proposition for prices that are normalized by the quasilinear good, but not necessarily for relative prices. Whether it holds for relative prices may depend on how commodities are represented. A doubling of commodity j in the aggregate supply vectors being compared will halve their prices and may overturn the comparative static result. Proposition 4 gives necessary and sufficient conditions for the comparative statics conclusion to hold for relative prices for all representations of commodities. The second condition in Proposition 4 is satisfied if all the cross partials of U are nonnegative. It is not satisfied, for example, if U is homogeneous and concave and $U(x_1, x_2, x_3, q) = q + (x_1x_2x_3)^{1/3} - .5(x_1x_2)^{1/2} + 40.2x_1 + 40.4x_2 + 40x_3$ in a neighborhood of $(.8, .4, .16, q)$.[6] Thus, depending on how commodities are represented, an increase in the supply of commodity j can lead to an increase in its relative price (but not an increase in its price relative to the price of the quasilinear good).

We index transformations of the commodity space by $r \in \mathbf{R}^{\ell}_{++}$, and let $\mathrm{I}(r)$ be the $\ell \times \ell$ diagonal matrix whose ith diagonal element is $1/r_i$, $i = 1, \ldots, \ell$. When the commodity space is transformed by $I(r)$ the aggregate utility function must also be transformed to $U^r : \mathbf{R}^{\ell}_{+} \rightarrow \mathbf{R}_{+}$ defined by $U^r(\mathrm{I}(r)x) = U(x)$. This transformation preserves preferences, and if U is concave and homogeneous, U^r is concave and homogeneous. We write x^r for $I(r)x$ and let $\overline{\rho}^r(x^r)$ be the market clearing prices that satisfy $F^{U^r}(\overline{\rho}^r(x), x) = x$ and $U^r(x^r) = \overline{\rho}^r(x^r) \bullet x^r$. The relative price of good j is $\rho^r(x^r)_j = \overline{\rho}^r(x^r)_j / |\overline{\rho}^r(x^r)|$. A proof of the following is in our working paper:

Proposition 4 (Comparative Statics for all Representations of Commodities) *Suppose there is a representative consumer with a utility function U that is differentiable, weakly monotone and homogeneous of degree 1 and suppose $x, y, \overline{\rho}(x), \overline{\rho}(y) \in \mathbf{R}^{\ell}_{++}$. Then the following are equivalent:*

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[6] To complete the example, one can either assume that the only aggregate supplies possible are in this neighborhood, or one can find a concave, homogeneous extension of U. To check that the second condition of Proposition 4 holds, differentiate U at $(.8, .4, .16, q)$.

1. *$\rho^r(x^r)_j \leq \rho^r(y^r)_j$ for all $r \in \mathbf{R}^\ell_{++}$*
2. *$\frac{\partial U(x)/\partial x_i}{\partial U(x)/\partial x_j} \geq \frac{\partial U(y)/\partial x_i}{\partial U(y)/\partial x_j}$ for all $i \neq j$.*

The next proposition shows that there is a law of supply regarding aggregate market values, if not prices. In the proposition suppose that $y < x$. Then y can be interpreted as part of the aggregate endowment x, and the experiment of comparing the prices at x to the prices at $x + y$ is to see what happens when the aggregate supply of y is doubled. The first two parts of the proposition state that the market value of y relative to the market value of the aggregate endowment x decreases (does not increase) if y is doubled. A relevant special case is when $y = ke^j$, $k > 0$, that is, y represents an increase in the aggregate endowment of the jth commodity. Then the relative value of the jth commodity decreases (does not increase) when its supply is doubled.

The proposition also implies a weak law of supply regarding prices rather than market values. We can interpret the second inequality in part 1 and part 3 for the case of quasilinear utility. When the aggregate supply of good j is increased its price relative to the numeraire falls (does not rise), and if commodities are initially represented such that the aggregate supplies are the same except for the quasilinear good, then the *relative* price of good j falls (does not rise) when its supply is increased.

Proposition 5 (Comparative Statics on Relative Value) *Suppose there is a representative consumer with a utility function U that is weakly monotone and homogeneous of degree 1 and that $x \in \mathbf{R}^\ell_{++}, y \in \mathbf{R}^\ell_{+}$. Then*

1. *$\frac{\overline{\rho}(x)\cdot y}{\overline{\rho}(x)\cdot x} \geq \frac{\overline{\rho}(x+y)\cdot y}{\overline{\rho}(x+y)\cdot x}$ and $\overline{\rho}(x)\cdot y \geq \overline{\rho}(x+y)\cdot y$*
2. *$\frac{\rho(x)\cdot y}{\rho(x)\cdot x} \geq \frac{\rho(x+y)\cdot y}{\rho(x+y)\cdot x}$*
3. *If $x = (1, 1, \ldots, q)$ and $\overline{\rho}(x)_\ell = \overline{\rho}(x+y)_\ell = 1$ (as when q is a quasilinear good), then $\rho(x)\cdot y \geq \rho(x+y)\cdot y$.*

Proof

(1) By Proposition 3(1) $\overline{\rho}(x)\cdot x \leq \overline{\rho}(x+y)\cdot x$ and $\overline{\rho}(x+y)\cdot(x+y) \leq \overline{\rho}(x)\cdot(x+y)$, hence $\overline{\rho}(x)\cdot y \geq \overline{\rho}(x+y)\cdot y$. This shows the second inequality. Further, we have $\overline{\rho}(x)\cdot x \leq \overline{\rho}(x+y)\cdot x$. Thus the first inequality also holds.

(2) Divide the numerator and denominator of the lefthand side of the first inequality in (1) by $|\overline{\rho}(x)|$ and the numerator and denominator of the righthand side by $|\overline{\rho}(x+y)|$. The first inequality follows.

(3) If $x = (1, 1, \ldots, q)$, by the first part of Proposition 3 $|\overline{\rho}(x)| - (1-q) \leq |\overline{\rho}(x+y)| - (1-q)$, hence $|\overline{\rho}(x)| \leq |\overline{\rho}(x+y)|$. Thus (3) follows from the second inequality in (1). $\square$

The Law of Supply in Market Games

A *market game* as defined by Shapley and Shubik [10] [11] is a game in characteristic function form where the set of payoffs available to each coalition arises from an exchange economy. Since agents in a market game are defined by their preferences and endowments, it is impossible to change the aggregate supply of commodities without changing the set of traders (hence the aggregate demand) or to change the set of traders without changing the aggregate supply of commodities. However one might think that if each commodity is disproportionately the endowment of one type of player, then increasing the proportional endowment of that commodity and increasing the proportional representation of its owners should have the same result: the price of the commodity and the utilities of its owners should fall when the proportional supply of either increases. This conjecture is true when agents have homogeneous, quasilinear utility functions.

In Proposition 6 we compare prices and values of endowments in two markets where the second market is like the first market, except that a player identical to player i is added.

Proposition 6 (Comparative Statics in Markets and Market Games) *Consider two markets,* $(\mathbf{M}, (U^j)_{j\in M}, (\omega^j)_{j\in M})$ *and* $(\mathbf{M}', (U^j)_{j\in M'}, (\omega^j)_{j\in M'})$*, where the agents in* $\mathbf{M}$ *are indexed* $j = 1, \ldots, m-1$ *and the agents in* $\mathbf{M}'$ *are indexed* $j = 1, \ldots, m$*. Suppose that agent* j*,* $j \leq m-1$*, has the same utility function and endowment in both markets, that* $U^m = U^i$*,* $\omega^m = \omega^i$ *for some* $i \in M$*, and that* $\omega = \sum_{j\in M} \omega^j \in \mathbf{R}^{\ell}_{++}$*. Suppose further that* U^j *is quasilinear in the* ℓ^{th} *good, weakly monotone, concave, and homogeneous of degree 1 for all* j*. Then*

1. *(One representative consumer for both markets)*
   There exists a quasilinear, weakly monotone, concave, homogeneous-of-degree-1 utility function U *such that the aggregate demands in the two markets are respectively* $F^U(p, \omega)$ *and* $F^U(p, \omega')$ *where* $\omega' = \sum_{j\in M'} \omega^j$*.*

2. *(The value of agent i's endowment falls when agent i is replicated, where the prices are normalized by letting the price of the quasilinear good equal one.)*
   $\frac{\overline{p}(\omega)\bullet\omega^i}{\overline{p}(\omega)\bullet\omega} \geq \frac{\overline{p}(\omega')\bullet\omega^i}{\overline{p}(\omega')\bullet\omega}$ *and* $\overline{p}(\omega)\bullet\omega^i \geq \overline{p}(\omega')\bullet\omega^i$

3. *(The value of agent i's endowment falls when agent i is replicated, where the prices are normalized in the simplex.)*
   $\frac{\rho(\omega)\bullet\omega^i}{\rho(\omega)\bullet\omega} \geq \frac{\rho(\omega')\bullet\omega^i}{\rho(\omega')\bullet\omega}$ *and if* $\omega = (1, 1, \ldots, q)$, $\rho(\omega)\bullet\omega^i \geq \rho(\omega')\bullet\omega^i$

4. *(The competitive payoff to agent i falls when agent i is replicated.)*
   $U^i(F^{U^i}(\rho(\omega), \omega^i)) \geq U^i(F^{U^i}(\rho(\omega'), \omega^i))$.

The proof is in our working paper. Part (1) follows from Lemma 2. Using Lemma 2, parts (2) and (3) restate Proposition 5, since $\omega' = \omega + \omega^i$. Part (4) follows from Proposition 1 applied to the games derived from the markets.

An application of particular interest might be the situation considered by Shapley and Shubik [11] in which each agent owns the entire social endowment of one commodity. Then it follows from the above proposition that if an agent is replicated, the relative value of that commodity (his endowment) falls.

Matching Models

In matching models there are two "sides" to the market, and only coalitions involving players from both sides can provide more utility for the parties than not making a match. For an arbitrary matching game Kelso and Crawford [5] and Crawford [1] compared its core with the core of another game in which more players were introduced on one side of the market (see also Roth and Sotomayor [6], [7], Theorems 2.5, 2.6, 8.17, 1.18). They discovered that in a certain sense the core payoffs of players on the augmented side decrease when they become more plentiful. Their result was not for all core payoffs, but only for those that favor one side of the market or the other.

Matching games satisfy the scale assumption (effective small groups) so that Proposition 2 applies by Lemma 1. We will use Proposition 2 to show that an increase in the number of players on one side of the market reduces the per-capita payoffs to such members. This result is 'weaker' than those mentioned above in that we do not claim that *all* players on one side of the market receive smaller payoffs. But it is 'stronger' than those mentioned above in that it applies to all payoffs in the cores of the two games.

For the following corollary to Proposition 2, let $\mathbf{W}$ be a *selection matrix*; i.e., a $T \times T$ nonnegative real diagonal matrix. The corollary follows by letting $\mathbf{N}' = \mathbf{N} + \mathbf{NW}$.

Corollary 1 *Suppose that* $\mathbf{u}$ *is in the core of* $(\mathbf{N}, V)$ *and* $\mathbf{u}'$ *is in the core of* $(\mathbf{N}+\mathbf{NW}, V)$*, where* $\mathbf{NW} \in \mathbf{Z}_+^T$*, and the two games exhaust blocking opportunities. Then* $\mathbf{u}' \cdot \mathbf{NW} \leq \mathbf{u} \cdot \mathbf{NW}$*.*

We now apply the corollary to matching games. For simplicity we will only discuss one-to-one matchings such as marriages, rather than many-to-one matching such as firms and workers, and we will restrict attention to matching models with side payments, since those correspond to the class of games discussed above.

We will assume there are T_f players on one side of the market, called the f side of the market (for 'female'), and T_m players on the other side, with a total of $T = T_f + T_m$. Each player will be a "type," and they will be ordered so that players $t = 1, ..., T_f$ are on the f side and $t = T_{f+1}, \ldots, T$ are on the m side. We want to know what happens to core payoffs if we augment the player set to $\mathbf{N}' = \mathbf{N} + \mathbf{NW}$, that is, we duplicate some players, and in particular if we duplicate some players on one side of the market. For this purpose we will assume that there are no conceivable types of players not represented in the player set $\mathbf{N}$. Thus all the coalitions in C are either singletons in $\mathbf{N}$ or coalitions in $\mathbf{N}$ consisting of one player from each side of the market. Thus by Lemma 1 the game $(\mathbf{N}, V)$ exhausts blocking opportunities, and we can use Corollary 1 to conclude the following remark.

Remark 2 *If a subset (or all) of the type-*f *players are replicated, then their per-capita core payoff falls. This follows from the corollary if we let* $\mathbf{W}$ *have zeros everywhere except in the first* T_f *rows of the principal diagonal.*

Concluding Remarks

The comparative statics of the core discussed above apply more broadly than to market games and matching games. In market games and matching games the core is nonempty, and we have therefore restricted our discussion to the core. More generally we show in our forthcoming paper that the comparative statics and monotonicity results extend also to approximate cores, as may be necessary to apply them to club economies or coalition production economies. Scotchmer [8] discusses how the comparative statics can be interpreted for price-taking equilibrium in club economies, and ES [3] discuss comparative statics of the "hedonic core," in which agents' core payoffs are decomposed as a linear function on their attributes. In comparing the cores or approximate cores of two games such that attribute j is scarcer in one than in the other, the hedonic payoff is greater in the game where it is scarcer. Here we have presented a special case in which an attribute is a 'type,' and each player has one unit of one attribute. In our forthcoming paper [3] we also discuss coalition production, where workers are compensated according to their skills, and club economies, where admissions prices reflect the externalities that a member inflicts on other members through his attributes.

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