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The core and hedonic core: reply to Wooders (2001), with counterexamples

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Abstract

In response to Wooders [Journal of Mathematical Economics 36 (2001) 295], I review the contributions of Engl and Scotchmer regarding the hedonic core and monotonicity [Journal of Mathematical Economics 26 (1996) 209], show how our contributions diverge from those previously in the literature, and highlight the importance of our assumptions by giving counterexamples, particularly to related results of Wooders.

Keywords: Core; Hedonic core; Monotonicity; Counterexamples

1. Introduction

Engl and Scotchmer (1996) prove two main results characterizing core payoffs in games where all players can be different in the sense of having different "attributes." First, in large games, core payoffs are approximated by hedonic payoffs to (or linear prices on) players' attributes. Second, a monotonicity (comparative statics) result applies: if a particular attribute is more heavily represented in one game than another (in a proportional sense), then its hedonic price should be lower (no higher) in the game where it is more heavily represented.

Wooders (2001) discusses many topics, some related to Engl and Scotchmer (1996). In this reply, I review our main contributions, draw some formal connections, and give a counterexample to Wooders' theorem in the main area of overlap. I discuss hedonic pricing in Section 2, and monotonicity in Sections 3 and 4.

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2. Hedonic pricing

Theorem 1 of Engl and Scotchmer (1996, circulated earlier in 1991, 1992) states conditions under which, in large games, core payoffs will be close to the value of players' attributes, evaluated at prices in the hedonic core. After we circulated our paper, Wooders sent us a series of papers that culminate in a similar result, Proposition 7 of Wooders (1993).[1]

Hedonic pricing is a strong result, and it only holds under restrictive conditions not satisfied by Wooders' Proposition 7. The main condition used by Engl and Scotchmer is that the payoffs available to coalitions of players are superadditive with respect to their attributes. This assumption has bite whenever the attributes are bundled into either commodities or players,[2] which is the case of interest.[3] The theorems we received from Wooders diverge from ours in various ways, but most importantly, in that she does not assume superadditivity in attributes.

The model below follows Engl and Scotchmer in defining a characteristic function on vector sums of attributes. However, for the purpose of stating Wooders (1993) Proposition 7, I adopt her notation and her assumption that there is a finite number of "types" of agents, and a finite number of endowment vectors of attributes.

Take as primitive a technology (Q, Λ), where Q is a positive integer (the number of elements in an attributes vector) and $\Lambda : Z_+^Q \to R_+$ is a characteristic function specifying the payoff to a bundle of attributes. Suppose there are T types of agents, and define the per-capita endowment of attributes for players of type t as $e^t \in Z_+^Q$. Let $E = \{e^1, e^2, \dots, e^T\}$. Let $f \in Z_+^T$ represent a *profile* of players, where f_t represents the number of agents of type-t, $t = 1, \dots, T$. Let $e(f) \equiv \sum_t f_t e^t$ be the aggregate attributes for a profile f. From the characteristic function Λ define another characteristic function $\psi : Z_+^T \to R_+$ by $\psi(f) \equiv$

[1] The stated objective of Wooders (2001) is to explain that she did not "duplicate" the results of Engl and Scotchmer. We first circulated our paper (June 1991) at a conference in Stony Brook. It was a complete paper, with complete proofs, and included a clear outline of an extension we were writing up. Our paper defined and developed the two topics mentioned above: the hedonic core and monotonicity. Two months later, Professor Wooders (1991) circulated a paper at another conference that included a four-page section titled "*Applications to Hedonic Pricing and Monotonicity*." The section neither states nor proves theorems, and Wooders admits on page 66 that her "discussion is not complete." Twelve months after that, in Wooders (1992c), "hedonic pricing" became the "attributes core." Six months after that, in Wooders (1993), the "attributes core" got a new definition.

[2] Engl and I point out (1996, p. 212) that "Superadditivity is sometimes thought to be an assumption without bite, since large coalitions can always subdivide into smaller ones, and that might be how the payoff of the large coalition is achieved. But in the context here, subdividing an attribute vector might mean that individuals must subdivide their time between different activities. That might or might not be possible."

[3] Bundling is inevitable when attributes are embodied in the players themselves, e.g. personal attributes such as Wooders' "ability to tell stories" (1993, p. 5). Wooders (1993, p. 15) explains that players in the game are "syndicates" of attributes. A syndicate is a group of players which has coalesced into one player, with its attributes treated as an indivisible unit.

Attributes can also be interpreted as attributes of commodities. The endowments in the example can be interpreted as attributes of three types of oranges. Type-1 oranges are sweet, type-2 oranges are bitter, and type-3 oranges are both. People like sweet oranges (which they can eat) or bitter oranges (to use in cooking), but oranges with the bundled attributes bitter/sweet have no use. As a consequence, the prices of oranges (and the utilities of their owners) cannot be described by a linear function on their attributes.

$\Lambda(e(f))$. Let (T, ψ) be the *derived pregame*, and let (f, ψ), $f \in Z_+^T$, be a *game*. The superadditive cover of ψ is $\psi^* : Z_+^T \to R_+$ defined by

$$\psi^*(f) = \max\left\{ \sum_{f^k \in P} \psi(f^k) | P \in \mathcal{P}(f) \right\}$$

where $\mathcal{P}(f)$ is the set of partitions of f, $\mathcal{P}(f) = \{\{f^k \in Z_+^T\} \mid \sum f^k = f\}$. Define the superadditive cover $\Lambda^* : Z_+^Q \to R_+$ analogously from Λ.

Using the notation $\|f\| = \sum_t f_t$, the ε-*core* of the game (f, ψ) is defined by[4]

$$C(f; \varepsilon) = \{x \in R^T : x \cdot f \le \psi^*(f),\ x \cdot s \ge \psi(s) - \varepsilon\|s\| \quad \text{for all } s \le f\}$$

For an "attributes game" (z, Λ), the *attributes* ε-*core* is similarly defined as[5]

$$C_a(z; \varepsilon) = \{p \in R^Q : p \cdot z \le \Lambda^*(z),\ p \cdot w \ge \Lambda(w) - \varepsilon\|w\| \quad \text{for all } w \le z\}$$

Wooders (1993), Proposition 7. *Let (Q, Λ) be a technology satisfying small-scale effectiveness,*[6] *let E be a finite set of endowments with $|E| = T$, and let (T, ψ) be the derived pregame. Let $\delta > 0$ and $\rho > 0$ be given positive real numbers. Then there is a positive real number ε^* and an integer $n_2(\delta, \rho, \varepsilon^*)$ with the following property: for each positive number ε with $\varepsilon \in (0, \varepsilon^*)$ and for each player profile f with $\|f\| > n_2(\delta, \rho, \varepsilon^*)$ and $f_t/\|f\| \ge \rho$ for each t, if both $C(f; \varepsilon)$ and $C_a(e(f); \varepsilon)$ are nonempty then a payoff x is in $C(f; \varepsilon)$ if and only if there is a price vector p in $C_a(e(f); \varepsilon)$ such that $|x_t - p \cdot e^t| < \delta$ for each $t = 1, \ldots, T$.*

Counterexample 1. Let there be three types of agents, $T = 3$, and two attributes, $Q = 2$. Let $e^1 = (2, 0)$, $e^2 = (0, 2)$, $e^3 = (1, 1)$. Let the characteristic function on attributes be defined by

$$\Lambda(z) = \begin{cases} z_1 & \text{if} \quad z_2 = 0 \\ z_2 & \text{if} \quad z_1 = 0 \\ \frac{1}{3}(z_1 + z_2) & \text{otherwise} \end{cases}$$

[4] Wooders (1993) gives two definitions of the ε-core. This is the definition of page 6. On page 17, she substitutes $\varepsilon\|e(s)\|$ for $\varepsilon\|s\|$. The distinction has no material consequence.

[5] This is the definition of Wooders (1993). Wooders (1992c) defines the ε-attributes core analogously to Engl's and Scotchmer's ε-hedonic core, as a payoff $p \in R^Q$ such that $p \cdot z \le \Lambda(z)$ and $p \cdot x \ge \Lambda(x) - \varepsilon\|x\|$ if $x \le z$. With this definition, when Λ is not superadditive, the ε-attributes core may easily be empty regardless of how large the game is.

[6] Given $\varepsilon > 0$, there is an integer $n_0(\varepsilon)$ such that for all endowments z, for some partition of z, $\{z^k\} \in \mathcal{P}(z)$, it holds that $\|z^k\| \le n_0(\varepsilon)$ for each z^k in $\{z^k\}$ and $\Lambda^*(z) - \sum_k \Lambda(z^k) < \varepsilon\|z\|$. This condition is satisfied in Counterexample 1 for $n_0(\varepsilon) = 1$ and every $\varepsilon > 0$.

The superadditive cover Λ^* is defined by $\Lambda^*(z) = z_1+z_2$. Both Λ and Λ^* are homogeneous of degree one.

From the characteristic function Λ on attributes, we can derive a characteristic function $\psi : Z^3_+ \to R_+$ on groups of players. Define ψ by $\psi(f) = \Lambda(e(f))$. Then

$$\psi(f) = \begin{cases} 2f_1 & \text{if} \quad f_2 = f_3 = 0 \\ 2f_2 & \text{if} \quad f_1 = f_3 = 0 \\ \frac{2}{3}(f_1 + f_2 + f_3) & \text{otherwise} \end{cases}$$

The superadditive cover is defined by $\psi^*(f) = 2f_1+2f_2+(2/3)f_3$. Even though $\psi(f) = \Lambda(e(f))$ for all f, it does not necessarily hold that $\psi^*(f)$ is equal to $\Lambda^*(e(f))$ for all f, e.g. $\psi^*(1, 1, 1) \not\equiv \Lambda^*(e(1, 1, 1))$.

We can now ask if the payoffs in the core of a game (f, ψ) are close to values of attributes, evaluated at prices in the attributes core of (z, Λ), where $z = e(f)$. For all $f \in Z^3_{++}$, and all $\varepsilon \geq 0$, $C(f; \varepsilon)$ is nonempty, since it contains the core payoff $x = (2, 2, (2/3))$. Core payoffs are achieved with each agent in a group of one, enjoying his own attributes. The core does not shrink as the game is enlarged. For all $z \in Z^2_{++}$, $C_a(z; 0) = \{(1, 1)\}$. If $p \in C_a(z; \varepsilon)$, then $p \geq (1 - \varepsilon, 1 - \varepsilon)$, and $p \cdot e^3 \geq 2 - 2\varepsilon$. Thus, $|x_3 - p \cdot e^3| \geq (4/3) - 2\varepsilon$, so the value of type-3 attributes will not be close to type-3 core payoff, 2/3, for small ε. □

Some gaps in the proof of Proposition 7 are given in Appendix B. The main gap in the reasoning arises from the assertion that $\psi^*(f) = \Lambda^*(e(f))$ for all f, which does not follow from $\psi(f) = \Lambda(e(f))$. Payoffs in the attributes core are defined with respect to Λ^*. However, such payoffs may not be feasible, or even approximately feasible, for a derived game (f, ψ). That is the source of the problem.

Given the counterexample, there is no need to dwell on the literature. Some time after hearing our work at Stony Brook, Wooders asked us to read her 1992a paper, and that is why we cited it. We found nothing there to suggest the "hedonic core" or "attributes core" (which came later), or any other attempt to derive structure for core payoffs when all players are different. On pages 297 and 299, Wooders (2001) quotes text from Engl and Scotchmer (1996) describing the 1992a paper as using "scale assumptions," which Engl and Scotchmer avoid.[7] As best I can figure out, Wooders (2001) now argues that her scale assumption is equivalent to assumptions used by Engl and Scotchmer (1996). But even without benefit of the counterexample, Wooders (2001) admits on page 296 that "it is an open question whether the convergence results of E&S imply those of this author or whether those of this author imply the E&S convergence results."

[7] The paper concerns games with a finite set of "types" of agents that stays fixed as the game is enlarged. Since all agents can be different in our model, these results do not apply. The most relevant theorems we could find in Wooders (1992a) are those for types games that are not necessarily replica games. These are Theorem 3.3 and Proposition A.2, which use the scale assumption "effective small groups." Wooders (1992a) writes on page 2, "we consider only core and approximate core payoffs with the equal treatment property.... The restriction is justified by the result, proven in the Appendix, that when small groups are effective...." Proposition A.2 in the appendix had no proof and no reference to one.

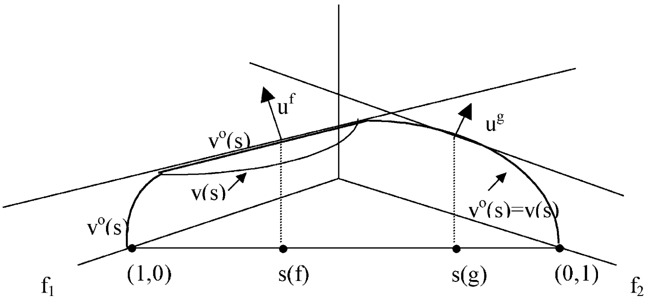

Fig. 1. Monotoxicity.

3. Monotonicity

The second contribution of Engl and Scotchmer concerns monotonicity or comparative statics of core payoffs (or hedonic core payoffs) in two games where the attributes are represented to different extents in aggregate. I first recapitulate the monotonicity result that Engl and Scotchmer generalize and extend.

Let Δ be the subset of the T-simplex consisting of the rational points. The simplex is shown in Fig. 1 for $T = 2$. For each vector $f \in Z_+^T$, let $s(f) = (f/\|f\|) \in \Delta$. Assuming that $\psi(f)/\|f\|$ has a bound that applies to all $f \in Z_+^T$, define $v : \Delta \to R_+$ by $v(s) = \sup_{r>0}(\psi(rs)/r)$. Following Scotchmer and Wooders (1988), define $v^o : \Delta \to R_+$

$$v^o(s) = \min_{u \in R_+^T} \{u \cdot s | u \cdot s' \geq v(s') \quad \text{for all } s' \in \Delta\} \tag{1}$$

The functions v and v^o are shown in Fig. 1, with $v \leq v^o$. These functions coincide if ψ is superadditive, as in Engl and Scotchmer (1996). The function v^o is the smallest concave function that is nowhere smaller than v.

A *bounding hyperplane* is $u \in R_+^T$ such that $u \cdot s \geq v^o(s)$, $s \in \Delta$, with equality at some $s \in \Delta$. Monotonicity is the mathematical expression of how bounding hyperplanes “tip” as the point of equality in Δ changes. To the best of my knowledge, this geometry, with its implications for monotonicity in games, was first pointed out by Scotchmer (1986), whom Wooders (2001) quotes on page 302.[8] Engl and Scotchmer cite Scotchmer and Wooders (1988) instead of Scotchmer (1986), as will this reply. I retired the 1986 paper when I agreed to develop the monotonicity idea jointly with Wooders.

[8] On page 302, Wooders (2001) quotes text from Scotchmer (1986) suggesting that the monotonicity results are derived from Wooders (1979a,b). Scotchmer (1986) cites Wooders (1979a). Nevertheless, in revisiting both papers, I can find no mention of, or even examples about, monotonicity or comparative statics. The 1979 papers are not cited by Scotchmer and Wooders (1988, 1989); indeed, Scotchmer and Wooders trace the geometry behind their proofs to Scotchmer (1986). Footnote 8 of Scotchmer and Wooders (1989) says “Other discussion of this geometry is in Scotchmer (1986),” and Footnote 8 of Scotchmer and Wooders (1988) says “This framework was introduced by Scotchmer (1986). . . ”

A hyperplane u represents a *payoff* for a game (f, ψ).[9] The payoff u can be *blocked* if $u \cdot f' < \psi(f')$ for some $f' \leq f$, which implies that $u \cdot s < v^o(s)$ for some $s \in \Delta$. If u is a bounding hyperplane, u cannot be blocked. The payoff u is *feasible* for (f, ψ) if there exists $\{f^k\} \in \mathcal{P}(f)$ such that $u \cdot f \leq \sum_k \psi(f^k)$. As above denote the core of the game (f, ψ) by $C(f; 0)$. Then $u \in C(f; 0)$ if u is feasible and cannot be blocked.

The monotonicity results hold when core payoffs can be represented by bounding hyperplanes to v^o. Consider two games (f, ψ) and (g, ψ) for which core payoffs are represented by u^f and u^g in Fig. 1. One can see that (2) hold, and (3) follows:

$$g \cdot u^g \leq g \cdot u^f, \qquad f \cdot u^f \leq f \cdot u^g \tag{2}$$

Monotonicity:

$$(g - f) \cdot (u^g - u^f) \leq 0 \tag{3}$$

The inequality (3) implies a law of scarcity or law of supply: if a single type of player becomes more plentiful, then the equal-treatment payoff to that type of player must decrease, or at least cannot increase.

The above intuition rests on the (so far) unfounded premise that core payoffs can be represented by bounding hyperplanes to v^o. Engl and Scotchmer defined a clean notion that guarantees this, which can also be extended to approximate monotonicity (see below). Let

$$\Omega_0 = \{u \in R_+^T | u \cdot s < v^o(s) \quad \text{for some } s \in \Delta\} \tag{4}$$

Following Engl and Scotchmer, a game (f, ψ) *exhausts blocking opportunities* if every $u \in \Omega_0$ can be blocked by some $\hat{f} \leq f$. If the game exhausts blocking opportunities, core payoffs are bounding hyperplanes to v^o, because all other feasible payoffs can be blocked. As a consequence, if the game is enlarged, the coalitions that are introduced do not have blocking power not already possessed by some coalition in the smaller game. For such a condition to make sense, the players that are introduced in enlarging the game must have something in common with players previously in the game. Here, they are drawn from the same set of "types." In Engl and Scotchmer (1996), they have attributes similar to those of other players, in the sense of being drawn from the same distribution.

Engl and Scotchmer (1997), Proposition 2. *If (f, ψ) and (g, ψ) exhaust blocking opportunities, and if u^f and u^g are, respectively, in their cores ($u^f \in C(f; 0)$ and $u^g \in C(g; 0)$), then the monotonicity inequality (3) holds.*

I now state three stronger assumptions that have been used to show monotonicity, (3). It is not the differences among A.1, A.2, and A.3 that I wish to emphasize, but their essential similarity. Each expresses the idea that all blocking power is possessed by coalitions in the game, as in Scotchmer and Wooders (1988), and blocking power cannot be increased by enlarging the game. Exhaustion of blocking opportunities subsumes A.1, A.2 and A.3, so their differences are of no particular interest.

[9] Such a payoff assumes equal treatment, which may not hold even in a types game. However, equal treatment will "almost" hold, since coalitions with similar attributes will receive similar payoffs in the hedonic core. See Engl and Scotchmer (1996), Corollary 4 to Theorem 1.

Assumption A.1 (Scotchmer and Wooders (1988), Proposition 2). The game (f, ψ) satisfies the condition that, for every bounding hyperplane u there exists $s \in \Delta$ and $r(s) \in Z_+$ such that $\psi(r(s)s) = r(s)v(s) = u \cdot r(s)s$, and for each s such that $v(s) = v^o(s)$ there exists $r(s)$ such that $\psi(r(s)s) = r(s)v^o(s)$ and $r(s)s \le f$.

Assumption A.2 (Wooders (1994), Proposition 4.1). There exists $B \in Z_+$ such that for each $\hat{f} \in Z_+^T$,

$$\max\left\{\sum_k w_k \psi(f^k) \,\middle|\, \sum_k w_k f^k = \hat{f},\ w_k \in R_+,\ f^k \in Z_+^T,\ \text{for each } k\right\} =$$

$$\max\left\{\sum_k w_k \psi(f^k) \,\middle|\, \sum_k w_k f^k = \hat{f}, w_k \in R_+,\ f^k \in Z_+^T,\ \|f^k\| < B \text{ for each } k\right\}$$

The game (f, ψ) satisfies $f \ge (B, B, B, \dots, B)$.

Assumption A.3 (Engl and Scotchmer (1997), Lemma 1). There exists $C = \{f^c \in Z_+^T\}$, including the T singleton profiles, such that $\psi(f) = \max\{\sum_i \psi(f^i)|,\ f^i \in C$, each i, $\sum_i f^i = f\}$. The game (f, ψ) satisfies $f^c \le f$ for each $f^c \in C$.

Proposition 1. *Each of A.1, A.2 and A.3 implies that the game (f, ψ) exhausts blocking opportunities.*

Proof. Suppose that $u \in \Omega_0$ so that $u \cdot s' < v^o(s')$ for some $s' \in \Delta$. We must show that u can be blocked under each of the three assumptions. Let $\rho > 1$ be such that ρu is a bounding hyperplane. If A.1 holds, there exists $s \in \Delta$ and $r(s) \in Z_+$ such that $r(s)s \le f$ and $\psi(r(s)s) = \rho u \cdot r(s)s > u \cdot r(s)s$, and u can be blocked by $r(s)s$. For A.2 and A.3, observe that $u \cdot s < v(s)$ for some s, so there exists $\hat{f} \in Z_+^T$ such that $u \cdot \hat{f} < \psi(\hat{f})$. Let $\sum_k w_k \psi(f^k)$ be the maximum in A.2 for $\hat{f}$. Then $u \cdot \hat{f} = u \cdot \sum_k w_k f^k < \psi(\hat{f}) \le \sum_k w_k \psi(f^k)$, so there exists f^k, $\|f^k\| < B$ and $f^k \le f$, such that $u \cdot f^k < \psi(f^k)$. Hence, u can be blocked by f^k. If A.3 holds, $\psi(\hat{f}) = \sum_k \psi(f^k)$ for some collection $\{f^k\} \in \mathcal{P}(\hat{f})$, where $f^k \in C$, each k. Thus, $u \cdot \hat{f} < \psi(\hat{f})$ implies that $u \cdot f^k < \psi(f^k)$ for some $f^k \in C$. Then u can be blocked by f^k. □

The following examples show that the A.1, A.2 and A.3 are not “nested,” and are special cases of exhaustion of blocking opportunities.

Example 1. Let $T = 2$, and let ψ be defined by

$$\psi(f) = \begin{cases} f_1 & \text{if} \quad f_1 > 0,\ f_2 = 0 \\ f_2 & \text{if} \quad f_2 > 0,\ f_1 = 0 \\ 12 & \text{if} \quad f_1 = f_2 = 3 \\ 0 & \text{otherwise} \end{cases}$$

The game $((3, 3), \psi)$ exhausts blocking opportunities and satisfies A.1, but not A.2. If we consider the superadditive cover, the game $((3, 3), \psi^*)$ also satisfies A.3.

Example 2. Let $T = 2$, and let ψ be defined by $\psi(f) = f_1 + f_2$. Every game (f, ψ) exhausts blocking opportunities. A game $((1, 1), \psi)$ satisfies A.2 and A.3, but not A.1.

Engl and Scotchmer (1997) attribute Proposition 2 to Scotchmer and Wooders (1988), even though Proposition 2 is more general than the monotonicity theorem of Scotchmer and Wooders (also Proposition 2). Scotchmer and Wooders realized that the condition underlying monotonicity is that all blocking power is possessed by groups contained in the games being compared; that is the essence of A.1, A.2 and A.3. However, on page 298, Wooders (2001) gives a partial quotation from Engl and Scotchmer, and takes issue with the attribution. The attribution is appropriate, especially because Scotchmer and Wooders (1988) was not published. It would have been opportunistic to take credit for the underlying idea based on an improvement to the formal statement of monotonicity, or a variant on the small-groups assumption.[10]

Of course there is always a question as to whether a particular result, definition or insight is novel enough to warrant attribution. That the role of small groups in monotonicity is nonobvious is evident in the following (incorrect) proposition of Wooders herself, where she asserts monotonicity without assuming that all blocking power is possessed by small groups. A counterexample follows.

Wooders (1992a), Proposition 4.1. *Let $[n, (T, \psi)]$ be a game determined by a pregame (T, ψ), and suppose that $[n, (T, \psi)]$ is totally balanced. Let f and g be subgames of n with the support of f equal to the support of g. Suppose that x is in the core of the subgame f and y is in the core of the subgame g. Then $(x - y) \cdot (f - g) \leq 0$.*

The text surrounding Proposition 4.1 refers to 1-homogeneity and concavity of ψ, which ensures that the cores of (f, ψ) and (g, ψ) are nonempty."Totally balanced" implies that every game (or "subgame") has a nonempty core. The support condition is satisfied if $f > 0$ and $g > 0$.

Homogeneity of the characteristic function seems like a plausible condition for exhaustion, since it can be interpreted to mean that there are no scale effects at all. However, this is deceptive. The condition that underlies monotonicity, namely exhaustion of blocking opportunities, is a condition on the *game*, and not a condition on the *characteristic function*. Homogeneity, as well as other conditions like per-capita boundedness and various primitive assumptions involving small groups, are conditions on the characteristic function, and do not directly reveal whether the requisite exhaustion hypothesis is satisfied.

[10] In the Introduction of Wooders (1992a), Wooders writes, "The monotonicity results are introduced in this paper." After I pointed out the errors and reminded her of Scotchmer and Wooders (1988), this claim was withdrawn. In the revised working paper Wooders (1992b), Proposition 4.1 became a monotonicity result based on the variant A.2 above, and appropriately attributed to Scotchmer and Wooders. However, in the published paper Wooders (1994), Proposition 4.1 is described as an "extension."

The following is a counterexample to Proposition 4.1.[11]

Counterexample 2. Let $T = 2$ and let ψ be defined by $\psi(f) = (f_1 f_2)^{1/2}$. The payoff $x = (1, 0)$ is in the core of $(f, \psi) = ((1, 1), \psi)$, and $y = (\sqrt{2} - 0.4, 0.2)$ is in the core of $(g, \psi) = ((1, 2), \psi)$. Thus, monotonicity does not hold: $(x - y) \cdot (f - g) = (1.4 - \sqrt{2}, -0.2) \cdot (0, -1) > 0$. Type-2 players receive a higher payoff in the game where they are more plentiful.

4. Asymptotic monotonicity and comparative statics

I now come to the main contribution of Engl and Scotchmer regarding monotonicity.

There is a kind of contradiction in the monotonicity theorem, Proposition 2 above. The monotonicity inequality (3) holds for payoffs in the cores of two games; hence the cores must be nonempty. But, in addition, the games must exhaust blocking opportunities. The latter requires that blocking opportunities are exhausted in bounded groups. These two requirements are often inconsistent: if blocking opportunities can be exhausted in finite games, then there is some notion of "efficient scale," which may lead to the result that the core is empty. On the other hand, if there are no such scale effects—e.g. the characteristic function is 1-homogeneous and concave, as in games derived from well-behaved exchange economies—then it may be impossible to satisfy exhaustion in finite games (see Counterexample 2 above). The main contribution of Engl and Scotchmer, as regards monotonicity, is to define a notion of approximate exhaustion (as above, a condition on the game, not the characteristic function), and to find the approximations to monotonicity that resolve this contradiction.

I will continue my exposition for the special case of "types," as it allows easy exposition.

Instead of discussing monotonicity on absolute numbers of players as in (3), Engl and Scotchmer discuss a notion of comparative statics on proportions. In Fig. 1, the compositions $s \in \Delta$ describe proportions of players, not absolute numbers. The monotonicity equation (3) becomes $(s(g) - s(f)) \cdot (u^g - u^f) \leq 0$ when expressed in proportions. However, that inequality does not allow us to conclude that if the proportion of type-j decreases, their core payoff increases (see Example 2 of Engl and Scotchmer (1996)). Instead, we must use the inequalities (2).

Following Engl and Scotchmer ((1996, Theorem 2(4)), or (1997), p. 541), say that[12] *the game* (f, ψ) *has proportionately more type-j players than* (g, ψ) if $s(f) = ks(g) +$

[11] Actually, the counterexample was already known to Scotchmer and Wooders (1988). Section 5 explains why a proposition like 4.1 could not be proved, and why finite games derived from exchange economies need not satisfy monotonicity.

We also pointed out, using the concave function v^o, that monotonicity holds if the characteristic function is 1-homogeneous and the player set is a continuum. This fact is stated formally by Wooders (1992b, 1994) as Proposition 4.3, arguing from a "limit utility function" U. But I show in the appendix how the limit utility function U is equivalent to v^o.

[12] This definition reappears in Proposition 2(B) of Kovalenkov and Wooders (1999): "If $(f^2/\|f^2\|) = (1 - \mu)(f^1/\|f^1\|) + \mu e^j$, $\mu \in (0, 1)$ (i.e. the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then $(x_j^2 - x_j^1) < (\varepsilon + \delta + \beta)(2 - \mu)/\mu$." Thus, they present a comparative statics result that is very similar to that of Engl and Scotchmer, reiterated as (7) below.

$ka\, s_j(g)\boldsymbol{e}^j$ for $a, k > 0$, where $\boldsymbol{e}^j$ is the jth unit vector. (These conditions imply that $k < 1$ and $k + ka > 1$.) The inequalities (2) can be written

$$ks(g) \cdot u^f \geq ks(g) \cdot u^g, \qquad s(f) \cdot u^g \geq s(f) \cdot u^f$$

Subtracting one from the other, $(s(f) - ks(g)) \cdot (u^f - u^g) \leq 0$, which implies $(ka)\, s_j(g)(u_j^f - u_j^g) \leq 0$, hence

$$u_j^f \leq u_j^g \tag{5}$$

Thus, if there are proportionately more type-j players in one game than another, type-j's core payoff must be either lower or the same as in the game with proportionately fewer type-j players.

But while (5) extends the monotonicity argument to a comparison of games where the numbers of all types of players are different, it still requires exhaustion of blocking opportunities, and it still refers to payoffs in the core. Engl and Scotchmer show that an approximate notion of comparative statics holds for changes in proportions when (i) blocking opportunities are only approximately exhausted, and (ii) the payoffs being compared are in the ε-core rather than core.

In Fig. 1, the intuition for the inequalities (2) only requires that the hyperplanes u^g and u^f are "close" to bounding hyperplanes. In that case, the hyperplanes will still either cross as shown, or will "almost" cross as shown. In order to define the requisite notion of "close," Engl and Scotchmer define

$$\Omega_\varepsilon = \{u \in R_+^T | u \cdot s < v^o(s) - \varepsilon \text{ for some } s \in \Delta,\ d(s, \partial\Delta) \geq \varepsilon\}$$

where $d(s, \partial\Delta)$ is the distance from the point $s \in \Delta$ to the boundary of Δ. Then the game (f, ψ) *ε-exhausts blocking opportunities* if every $u \in \Omega_\varepsilon$ can be blocked by some $\hat{f} \leq f$. For arbitrary $\varepsilon > 0$, sufficiently large games will ε-exhaust blocking opportunities.

The following is an informal statement of our 1996 comparative statics theorem (Theorems 2 and 3), rendered for the case of "types" instead of attributes games. Suppose the game (f, ψ) has proportionately more type-j players than (g, ψ). Then, given $\gamma, \psi > 0$ and an assumption that assures sufficient overlap among the attributes of players in the games, there exists $\varepsilon > 0$, such that if both games ε-exhaust blocking opportunities, and if $u^f \in C(f; \varepsilon)$, $u^g \in C(g, \varepsilon)$, then

$$u^f \cdot s(g) \geq u^g \cdot s(g) - \psi, \qquad u^g \cdot s(f) \geq u^f \cdot s(f) - \psi \tag{6}$$

$$u_j^f \leq u_j^g + \gamma \tag{7}$$

The inequalities (6) also yield "asymptotic monotonicity," $(s(f) - s(g)) \cdot (u^f - u^g) \leq 2\psi$, but we do not stress it because it does not imply (7).

Engl and Scotchmer (1991, 1992, 1996) discuss a more complicated class of games than the types games mentioned here, ones in which all players can be different in the sense of having different attributes, and ones in which the attributes in large games can be drawn randomly. The fact that core payoffs can be described by hedonic values of attributes allows us to apply the comparative statics conclusion to prices on attributes.

In our 1997 paper, we apply these ideas to market games, showing, for example, that if the aggregate endowment of some commodity increases, its price will decrease (not increase), and if a player is duplicated, the value of his endowment, evaluated at equilibrium prices, will decrease, which also causes his utility to decrease.

5. Conclusion

There is no issue of priority with respect to the contributions of Engl and Scotchmer outlined above. Wooders (2001) also makes many other remarks for which a reply in a scientific journal seems inappropriate. They are implicitly addressed by the above account. For the record, I reject them.

Appendix A. The limit function U and the function v^o

A limit "utility function" U is used in the proof of Proposition 7. In addition to defining it, I show that it is the same function as the function v^o above, extended homogeneously to the nonnegative quadrant.

The limit function U is the continuous extension to R^T_+ of a function $\hat{U}$: $\hat{R}^T_+ \to R_+$, defined as follows, where $\hat{R}^T_+$ are the rational points in R^T_+. For each $f \in \hat{R}^T_+$ choose $r(f)$ so that $r(f)f$ is a vector of integers. Then let

$$\hat{U}(f) = \lim_{\nu\to\infty} \frac{\psi^*(\nu r(f)f)}{\nu r(f)}$$

Lemma 1. $\hat{U}(s) = v^o(s)$ *for all* $s \in \Delta$.

Proof. I show that at each $s \in \Delta$, $\hat{U}(s)$ is neither larger nor smaller than $v^o(s)$.

Suppose that $v^o(s) > \hat{U}(s)$. Since $\hat{U}$ is concave and nonnegative, there exists $u \in R^T_+$ such that $v^o(s) > \hat{U}(s) = u \cdot s$ and $u \cdot \zeta \geq \hat{U}(\zeta)$ for all $\zeta \in \Delta$. From the definitions of v and $\hat{U}$, it holds that for each $s \in \Delta$, $\hat{U}(s) \geq v(s)$. Hence, $v^o(s) > u \cdot s$ and $u \cdot \zeta \geq v(\zeta)$ for all $\zeta \in \Delta$. This contradicts the definition of v^o.

Suppose that $v^o(s) < \hat{U}(s)$ for some $s \in \Delta$. Choose a positive number ε such that $v^o(s) < \hat{U}(s) - \varepsilon$. Using the definition of v^o, for some $u \in R^T_+$, $u \cdot s = v^o(s)$ and $u \cdot \zeta \geq v(\zeta)$ for all $\zeta \in \Delta$. Then it follows from the definitions of $\hat{U}$ and ψ^* that there exists (large) $\nu > 0$ and $\{f^k\} \in \mathcal{P}(\nu r(s)s)$ such that

$$\hat{U}(s) - \varepsilon < \sum_k \frac{\psi(f^k)}{\nu r(s)} = \sum_k \frac{|f^k|}{\nu r(s)} \frac{\psi(f^k)}{|f^k|} \leq \sum_k \frac{|f^k|}{\nu r(s)} v(s(f^k))$$
$$\leq \sum_k \frac{|f^k|}{\nu r(s)} u \cdot s(f^k) = u \cdot \sum_k \frac{|f^k|}{\nu r(s)} s(f^k) = u \cdot \sum_k \frac{f^k}{\nu r(s)} = u \cdot s = v^o(s)$$

but since $v^o(s) < \hat{U}(s) - \varepsilon$, this is a contradiction. □

Thus, Wooders' function $\hat{U}$ is the 1-homogeneous extension of Scotchmer's and Wooders' v^o. Both can be extended continuously to a concave, homogeneous (and superadditive) function U on all of R_+^T. Wooders analogously defines a limiting utility function W from the characteristic function Λ, using its superadditive cover Λ^*. The function W is Engl and Scotchmer's (1991, 1992, 1996) $\hat{V}$.

Appendix B. Gaps in the proof of Proposition 7

On pages 17 and 18, Wooders (1993) asserts without proof that

$$\{x \in R_+^T | x \cdot f \leq \psi^*(f) \text{ and } x \cdot s \geq \psi(s) - \varepsilon \| e(s) \| \text{ for all } s \leq f\}$$
$$= \{x \in R_+^T | x \cdot f \leq \Lambda^*(e(f)), \text{ and } x \cdot s \geq \Lambda(e(s)) - \varepsilon \| e(s) \| \text{ for all } s \leq f\}$$

But, as shown by Counterexample 1, even if $\psi(f) = \Lambda(e(f))$, it might not hold that $\psi^*(f) = \Lambda^*(e(f))$.

Proposition 5 asserts without proof that if ψ is derived from Λ as described above, so that $\psi(f) = \Lambda(e(f))$, then $U(f) = W(e(f))$. But, as shown by the example, it might not hold that $\psi^*(f) = \Lambda^*(e(f))$. In the example, $U(f) = \psi^*(f)$ and $W(e(f)) = \Lambda^*(e(f))$ on the domains where both are defined, and thus $U(f)$ might not be equal to $W(e(f))$.

The first paragraph of the proof of Proposition 7 asserts without proof that for any $\varepsilon > 0$, "if p is in $C_a(e(f); \varepsilon)$ then $\pi \in R_+^T$ is in $C(f; \varepsilon)$ where π is given by $\pi_t = p \cdot e^t$ for each t." The assertion does not hold in the example, since the payoffs $(p \cdot e^1, p \cdot e^2, p \cdot e^3)$ need not be feasible for the game (f, ψ).

On page 20, Proposition 6 is used in the proof of Proposition 7. However, the proof of Proposition 6 relies on an analogy that is not apt for Proposition 7, and not reflected in its hypotheses. Proposition 6 asserts that "$p \in P(e(f))$ if and only if there is $\pi \in \Pi(f)$ such that $p \cdot e^t = \pi_t$ for each t with $f_t > 0$." $\Pi(f)$ is the set of subgradients of U at f. For each $z \in R_+^Q$, $P(z)$ is "the set of competitive prices relative to the endowment z" in an economy "where each participant owns one unit of one attribute and the utility function of each participant is W. It is convenient to call $P(\cdot)$ the competitive price correspondence for (Q, W)." The proof of Proposition 6 uses the fact that "the competitive equilibrium payoffs of an economy with endowment $e(f)$ are in the core of" the derived game. However, in the economies of Proposition 7, nothing restricts the endowment vectors in E to a single attribute. Indeed, to make such an assumption would undermine the relevance of Proposition 7 for the question addressed by Engl and Scotchmer, or the economies apparently of interest to Wooders (see Footnote 3). In the example, which satisfies the hypotheses of Proposition 7, W is defined by $W(z) = z_1 + z_2$, U is defined by $U(f) = 2f_1 + 2f_2 + (2/3)f_3$, $P(\cdot) = \{(1, 1)\}$ and $\Pi(\cdot) = \{(2, 2, (2/3))\}$. For type-3, the conclusion of Proposition 6 does not hold, since $p \cdot e^3 = 2 \not\equiv (2/3) = \pi_3$.

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