The core and the hedonic core: 
Equivalence and comparative statics

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Abstract

In large cooperative games core payoffs can be decomposed as a linear function on players' attributes which we call a 'hedonic payoff', provided that the worth of each coalition depends only on the sum of its members' attributes and payoffs are superadditive in attributes. If two large finite games weight a particular attribute differently, then the hedonic payoff to that attribute is larger (no smaller) in the game that gives it less weight. The hedonic payoff can be interpreted as a competitive price function e.g. an anonymous wage function in coalition production economies and an anonymous system of admissions prices in club economies.

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1. Introduction

There is a long-standing conjecture in economics that the competitive prices of commodities should depend only on their economically relevant attributes. Commodities with similar attributes should have similar prices. This idea was initially proposed by Lancaster (1971), and has been widely applied to land and labor.

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'Hedonic price functions', as they are called, are often assumed for purposes of estimation to be linear, but we know of only one justification for this, given by Jones (1988) for a particular restriction on preferences.

In this paper we price the attributes of individuals instead of commodities, and show that in large games with transferable utility core payoffs can be decomposed as a linear function of players' attributes which we call a hedonic payoff. Our main assumptions are that the payoff of a coalition depends only on the total attributes of the coalition's members, and not, for example, on the number of members, and that payoffs are superadditive in attributes.

We represent feasible payoffs with a characteristic function on the domain of attributes. Unless this characteristic function is homogeneous, the core might be empty. Our approximation result is therefore for the e-core (Shapley and Shubik, 1966), and applies also to the core. We show that a linear function on attributes approximates payoffs in the e-core except possibly for coalitions that represent a small fraction of the player set. This fraction can be arbitrarily small for a sufficiently large game.

The equivalence of the core and the hedonic core has several consequences. First, while core payoffs are not unique, they are almost unique in the sense that any two core payoffs are close.

Second, the natural metric on the attributes space together with the linear hedonic payoff permits us to say that when attributes are close, payoffs are close.

Third, the linear function permits us to say how core payoffs depend on the relative importance of different attributes in the player set: provided two games 'exhaust blocking opportunities', if an attribute is represented more heavily in one game than in the other, then the hedonic payoff to that attribute is lower.

Fourth, the hedonic payoff can be interpreted as a competitive price system in coalition production economies and club economies.

Some of these conclusions are immediate in the continuum framework of Aumann and Shapley (1974), who assume that there are no scale effects in feasible payoffs. Our approximation theorem verifies that for purposes of characterizing the core, large finite games are similar to continuum games, and for this the homogeneity assumption is unnecessary. The only assumptions on feasible payoffs that we use are superadditivity and finiteness of per capita payoffs. This is in contrast to the literature on large games (see, for example, Wooders and Zame, 1984; Scotchmer and Wooders, 1988; and other papers, summarized in Wooders, 1992) which uses a 'scale' assumption to bound the size of coalitions with blocking power.

In games derived from exchange economies, it is known that the core converges to competitive payoffs. See Anderson (1992) for a summary of such results. We show that a similar interpretation applies here: the hedonic payoffs can be interpreted as competitive prices in coalition production economies and club economies, and thus our approximation theorem can be interpreted as a core convergence theorem. In coalition production economies the hedonic payoff
describes wages. Individuals' wages differ because their attributes differ, but the wage function is anonymous in that the price of each attribute is the same for all workers.

In club economies, individuals pay admission prices to join groups, and members of groups confer positive and negative externalities on each other. The original example of a positive externality is that members of the group can jointly contribute to a public good. The usual example of a negative externality is crowding.

Individuals with different characteristics will typically impose different externalities on each other in a club. Members may care whether the other members are well educated, well mannered or generous. We use the hedonic payoff to show how admissions prices to clubs reflect members' attributes. The anonymity of the hedonic payoff is inherited by admissions prices, which are anonymous in that each player pays the same prices for the attributes he contributes (externalities he confers), but non-anonymous in that players pay different admissions prices because they have different attributes.

The paper is organized as follows. In Section 2 we define the hedonic core and ε-hedonic core, and describe an example showing that the core and hedonic core might not coincide for small player sets. In Section 3 we discuss the approximation of the core by the hedonic core. In Section 4 we discuss the comparative statics of the hedonic core. Section 5 shows how the hedonic price function can be interpreted as wages in a coalition production model and as admissions prices in club economies, and the approximation theorem can be interpreted as a core convergence theorem.

2. The core and the hedonic core

We consider a finite set $N$ of players. Each player $i \in N$ is characterized by a vector of attributes $A^i \in \mathbb{R}_+^T \setminus \{0\}$. The $r$th coordinate of player $i$'s attribute vector, $A^i_r$, represents the amount of the $r$th attribute, such as a work skill or resource endowment possessed by player $i$. A standard unit vector in $\mathbb{R}^T$, representing one unit of a particular attribute, can be thought of as a 'type' of player.

A coalition $S$ is a non-empty subset of $N$. The attributes of a coalition $S$ are simply the sum of its members' attributes, $A^S = \sum_{i \in S} A^i$. The composition of a player's or coalition's attributes is $a^S = A^S / \| A^S \|$, where $\| \cdot \|$ is the 1-norm in $\mathbb{R}^T$, i.e. $\| A^S \| = \sum_r A^S_r$. A composition is in the simplex $\Delta = \{ a \in \mathbb{R}_+^T \mid \sum_i a_i = 1 \}$. We denote the interior of the simplex by $\text{int} \Delta = \{ a \in \Delta \mid a_i > 0 \text{ for all } i \}$.

Our premise is that the utilities achievable by a coalition depend only on the coalition's attributes, and not otherwise on the number or identities of its mem-

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1 As usual, we use $|N|$ to denote the cardinality of $N$. 
bers. Thus, we assume that feasible utilities are described by a function $V: R_+^N \setminus \{0\} \to R_+$, where $V(A)$ is the total utility or profit available to a coalition or individual with attributes $A$. We assume that $V$ is superadditive: $^2$ $V(A + A') \geq V(A) + V(A')$.

A game is an ordered triple, $(N, (A^i)_{i \in N}, V)$, which we will henceforth abbreviate as $(N, V)$, where $N$ is a player set, $(A^i)_{i \in N}$ gives the players' attributes, and $V$ is a payoff function as above. A payoff is a vector $U = (U^i)_{i \in N}$ in $R_+^{1 \times N}$ such that $\sum_{i \in N} U^i \leq V(A^N)$. The payoff to player $i$ under $U$ is simply $U^i$. A coalition $S$ can block $U$ if $\sum_{i \in S} U^i < V(A^S)$. A payoff $U$ is in the core of $(N, V)$ if no coalition can block it.

Since the core may be empty, we will also discuss $\epsilon$-cores, originally defined by Shapley and Shubik (1966). $^3$ A coalition $S$ can $\epsilon$-block a payoff $U$ if $\sum_{i \in S} U^i < V(A^S) - \epsilon \cdot |A^S|$. A payoff $U$ is in the $\epsilon$-core of $(N, V)$ if no coalition can $\epsilon$-block it. We denote the $\epsilon$-core of $(N, V)$ by $C_\epsilon(N, V)$ for $\epsilon \geq 0$, so that $C_0(N, V)$ is the core.

A hedonic payoff for $(N, V)$ is a vector $w \in R^T$ such that $w \cdot A^N \leq V(A^N)$. The vector $w$ is called a hedonic payoff because it represents payoffs to attributes rather than to players. The payoff to player $i$ under $w$ is $w \cdot A^i$. A coalition $S$ can block $w$ if $w \cdot A^S < V(A^S)$, and can $\epsilon$-block $w$ if $w \cdot A^S < V(A^S) - \epsilon \cdot |A^S|$. The $\epsilon$-hedonic core of $(N, V)$, denoted by $C^H_\epsilon(N, V)$ for $\epsilon \geq 0$, is the set of all hedonic payoffs that cannot be $\epsilon$-blocked by any coalition. We refer to $C^H_0(N, V)$ as the hedonic core.

It is easy to verify that for $\epsilon > 0$, and for a sufficiently large game that depends on $\epsilon$, the $\epsilon$-hedonic core is non-empty. (See the appendix, Proposition A.1.) $^4$ It is also easy to verify that if $w$ is in the $\epsilon$-hedonic core, then the payoffs $(w \cdot A^i)_{i \in N}$ are in the $\epsilon$-core, which is consequently non-empty. $^5$ The theorem in Section 3 addresses the converse; whether every payoff in the $\epsilon$-core corresponds to a payoff in the $\epsilon$-hedonic core.

Fig. 1 shows an example of a game with three players and two attributes. The simplex $\Delta$ is the line above which the concave function $V$ is drawn. For purposes

$^2$ Superadditivity is sometimes thought to be an assumption without bite, since large coalitions can always subdivide into smaller ones, and that might be how the payoff of the large coalition is achieved. But in the context here, subdividing an attribute vector might mean that individuals must subdivide their time between different activities. This might or might not be possible.

$^3$ Their definition is that no coalition can increase its utility by more than $\epsilon$ per capita. We define it differently for convenience, but under our assumptions the two definitions are essentially equivalent.

$^4$ The argument in Proposition A.1 differs from other such arguments in that it relies on the uniform convergence of $V(ra)/r$ to $v$ discussed below and avoids the discussion of balancedness.

$^5$ In general it is harder to prove non-emptiness of the $\epsilon$-core than under our assumptions, e.g. the proof of Wooders and Zame (1984) is more complicated because they do not assume that the worth of a coalition depends only on the sum of attributes. The authors use the term 'pregame' to describe the primitives, the technology and attributes space, from which the large game is taken. In this paper these are $V$ and $R^S_\epsilon$. 

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of representing our arguments in diagrams such as Fig. 1, it is convenient to
define, for fixed payoffs \((U^i)_{i \in N}\), the total payoff to a coalition \(S\) by \(U^S = \sum_{i \in S} U^i\), and its payoff normalized by its attributes as \(u^S = U^S / |A^S|\).

The following lemma is useful in proving the propositions and in illustrating
them. It can easily be verified.

**Lemma 1.** Let \(U\) be a payoff. If \(S_1 \cup S_2 = S\), then \(u^S = \lambda u^{S_1} + (1 - \lambda) u^{S_2}\) and \(a^S = \lambda a^{S_1} + (1 - \lambda) a^{S_2}\), where \(\lambda = |A^{S_1}| / |A^S|\).

In Fig. 1 the compositions of the three players' attributes are labeled \(a^i = A^i / |A^i|, i = 1, 2, 3\). The composition of the game is \(a^N = (A^1 + A^2 - A^3) / |A^1 + A^2 + A^3|\), and it is a convex combination of \(a^1, a^2\) and \(a^3\). We assume for this example that \(V\) is homogeneous, and together with superadditivity, this implies that \(V\) is concave as drawn. On the simplex the function \(V\) represents feasible payoffs per unit attribute that a coalition possesses, e.g. singleton coalitions can achieve \(V(a^i) / |A^i| = V(a^i), i = 1, 2, 3\). Similarly, a two-person coalition \(\{2, 3\}\) can achieve a total of \(V(a^{(2,3)})\) per unit attribute they possess. The linear function \(w\) represents a payoff in the hedonic core, and player \(i\)'s payoff in the hedonic core, normalized by the size of his attributes, is the point on the line above his composition \(a^i\), namely \(w \cdot a^i\).

Fig. 1 also shows payoffs in the core, normalized by attributes \(u^i = U^i / |A^i|, i = 1, 2, 3\). These core payoffs are not hedonic payoffs because they do not lie on a line. They are, however, in the core. No singleton coalition could block because each \(u^i\) lies above \(V\). The coalition of the whole cannot block because their payoff, \(u^{(1,2,3)} = w \cdot a^{(1,2,3)}\), is no smaller than \(V(a^{(1,2,3)})\). By Lemma 1 the coalition \(\{1, 3\}\) could block only if the line that connects \(u^1\) with \(u^3\) is below \(V(a^{(1,3)})\) at the composition \(a^{(1,3)}\), since \(u^{(1,3)}\) lies directly above \(a^{(1,3)}\). As drawn, no combination of two players could block.
3. Equivalence of the approximate core and approximate hedonic core

The following example shows that in general we cannot approximate the \( \epsilon \)-core uniformly (for all players) by a linear function. This example shows that the failure to converge uniformly is due to the more stringent blocking condition of the \( \epsilon \)-core, and not to the fact that \( V \) may not be homogeneous.

**Example 1.** Suppose there is only one attribute, and every player possesses it in the same amount, say \( A^i = 1 \). Then the payoff to a coalition \( S \) depends only on the number of players in \( S \), which we will denote by \( n^S \). We assume payoffs are given by a superadditive function \( V: R_+ \setminus \{0\} \rightarrow R_+ \) such that \( \sup_{n > 0} V(n)/n = \nu < \infty \), e.g. a technology with an efficient scale such as shown in Fig. 2. Then \( \lim_{n \rightarrow \infty} V(n)/n = \nu \) by Lemma A.2 in the appendix. We consider a sequence of games \( (N^n, V) \), where \( |N^n| = n \). A payoff \( U \) is in the \( \epsilon \)-core of \( (N^n, V) \) if for all coalitions \( S \), \( \sum_{i \in S} U^i \geq V(n^S) - n^S \epsilon \). Let

\[
U^i = \begin{cases} 
\frac{\epsilon}{2} (n - 1) & \text{for } i = 1, \\
\frac{1}{n - 1} \left[ V(n) - \frac{\epsilon}{2} (n - 1) \right] & \text{for } i = 2, \ldots, n.
\end{cases}
\]

We claim that \( U \in C_\epsilon(N^n, V) \) if \( \epsilon > 0 \) and \( n \) is sufficiently large. The vector \( U \), is a payoff since \( \sum_{i \in N} U^i = V(n) = V(N^n) \). To see that no coalition can \( \epsilon \)-block \( U \), we first consider coalitions not containing agent 1. Blocking by such a coalition would require \( V(n^S) > \sum_{i \in S} U^i + n^S \epsilon = [n^S/(n - 1)] [V(n) - (n - 1) \epsilon/2] + n^S \epsilon \), which implies that \( V(n^S)/n^S > V(n)/(n - 1) + \epsilon/2 \), and therefore that \( V(n^S)/n^S > [V(n)/n] [n/(n - 1)] + \epsilon/2 \). However, since \( [V(n)/n] [n/(n - 1)] \rightarrow \nu \) as \( n \) becomes large, \( S \) cannot \( \epsilon \)-block. A coalition of size \( n^S \) containing player 1 also cannot \( \epsilon \)-block when \( n \) is sufficiently large because the payoff of the coalition is even larger when \( S \) contains player 1, which makes it even harder to \( \epsilon \)-block.

Since a hedonic payoff would treat the players equally, and since \( U^1 \rightarrow \infty \) as \( n \rightarrow \infty \), while \( U^1 \rightarrow \nu - \epsilon/2 \) for all other players, the \( \epsilon \)-core payoffs do not converge uniformly to hedonic core payoffs.
Because this example shows that in general we cannot obtain uniform convergence for the $\varepsilon$-core, the next theorem shows 'almost' uniform convergence: a linear function of attributes closely approximates payoffs in the $\varepsilon$-core for all coalitions except those that are a small fraction of the player set. The fraction can be arbitrarily small provided the player set is large enough. In the example, the fraction of the player set represented by player 1 is not fixed. It becomes small as the player set becomes large. For the case that $V$ is homogeneous so that the core is non-empty, similar methods show that under an additional condition convergence of the core to the hedonic core is uniform (Engl, 1993).

It will be convenient to define a function that represents the supremum of average payoffs as the total amount of a coalition's attributes varies, holding the composition $a \in \Delta$ fixed. Thus we define $v: \Delta \rightarrow \mathbb{R}_+$ by

$$v(a) = \sup_{r > 0} \frac{V(ra)}{r}.$$

We assume that $v$ is not identically infinite everywhere on the interior of the simplex. The function $v$ is concave and $V(ar)/r$ converges to $v$ uniformly on compact sets in the interior of the simplex. (See Lemmas A.1 and A.2 in the appendix.)

In a large game $(N, V)$, a payoff $w$ in the hedonic core can be represented as a hyperplane that is 'close' to a supporting hyperplane $v$ at $a^N$. The hyperplane $w$ cannot lie below $V$ at any $A^S$ for any coalition $S$. If the game is large, then $V(A^N)/|A^N|$ is close to $v(a^N)$, and similarly, if a coalition $S$ is large, then $V(A^S)/|A^S|$ is close to $v(a^S)$. Therefore, the hyperplane $w$ cannot lie far below $v$ at $a^N$ or at $a^S$ if $S$ is large. If the compositions of players' attributes are dispersed, then $a^N$ will be in the interior of their convex hull, and $a^N$ will be densely surrounded by compositions $a^S$, where each $S^i$ is a large coalition. Therefore, if $w$ were not close to a supporting hyperplane at $a^N$, then some such coalition could block it.

The main results of this section are corollaries to the following theorem. In Corollary 1 we consider games in which the players' attributes are drawn independently from a distribution $F$. We will interpret $a^F$ as the mean of the distribution $F$ so that the second condition of the theorem is satisfied with high probability. In Corollary 2 we consider replication games. We will interpret $a^F$ as the composition of an initial game. All replications have the same composition, so that the second condition of the theorem is satisfied trivially. In Corollary 3 we state the implication that any two payoff vectors in the core are close to each other, and in Corollary 4 we show that Theorem 1 applies also when $V$ is defined on $Z^T$, the integral points in $\mathbb{R}_+^T$, rather than on all of $\mathbb{R}_+^T$. Corollary 4 addresses the 'types' case where each person has one unit of one attribute, and feasible coalitions are represented by vectors of integers.
Theorem 1. Let \((N, V)\) be a game, and suppose that \(V \geq 0\), there exist \(\Delta, A\) such that \(0 \leq \Delta \leq |A| \leq A\) for all players \(i \in N\), and that \(v\) is differentiable at \(a^F \in \text{int} \ \Delta\). Let \(\alpha \in (0, 1], \delta > 0\) be given. Then there exist \(n, r, \varepsilon_0 > 0\) such that if \(^6\)

\[
\begin{align*}
(1) & \ |N| \geq n, \\
(2) & \ |a^N - a^F| \leq r, \\
(3) & \ \varepsilon \in [0, \varepsilon_0], \text{ and} \\
(4) & \ |S| \geq \alpha |N|,
\end{align*}
\]

then

\[U \in C_e(N, V) \text{ and } we C^H_e(N, V) \Rightarrow |w \cdot A^S - U^S| < \delta |A^S|.
\]

The assumption that \(v\) is differentiable at \(a^F\), which implies that \(v\) has a unique supporting hyperplane at \(a^F\), is less restrictive than it appears, since \(v\), being concave, is differentiable almost everywhere. For example, in the coalition production economy described in Section 5, where \(V\) is a production technology, this assumption is satisfied provided the production technology is differentiable, homogeneous and concave. Differentiability of \(v\) is implied by differentiability of \(V\), provided \(\max_{r > 0} V(ar)/r\) exists and the maximizers are given by a differentiable function, say \(r^*(a)\). This describes the case that for each \(a \in \Delta\) there is an efficient scale, and the efficient scale is a differentiable function of the composition.

We now form a large game by drawing players' attributes independently according to a distribution \(F\) on \(R^T_+\). \(^7\) Let \(a^F = E F A^I/|E F A^I|\), where \(E F A^I\) is the mean of a random draw from \(F\). We assume that \(F\) is non-degenerate in the sense that \(a^F\) is in the interior of \(R^T_+\), and also that it has bounded support. For example, \(F\) might have full support on a set \(\{A \in R^T_+ | 0 < A \leq |A| \leq A\}\) or on a finite set of points, so that players have 'types'. Corollary 1 uses the fact that the condition \(|a^N - a^F| \leq r\) will hold with high probability for a sufficiently large player set with attributes drawn independently according to \(F\), and therefore by stating the theorem probabilistically, we do not need the second condition of the theorem.

Corollary 1. Suppose that the attributes of the players in a game \((N, V)\) are drawn independently from a distribution \(F\) with support in \(\{A \in R^T_+ | 0 < A \leq |A| \leq \bar{A}\}\) for some \(\Delta\) and \(\Delta\), and that \(v\) is differentiable at \(a^F \in \text{int} \ \Delta\). Let \(\alpha \in (0, 1], \delta, \theta > 0\) be given. Then there exist \(n, \varepsilon_0 > 0\) such that if

\[
\begin{align*}
(1) & \ |N| \geq n, \\
(2) & \ \varepsilon \in [0, \varepsilon_0]
\end{align*}
\]

\(^6\) We denote the usual Euclidean norm in \(R^T\) by \(|\cdot|\).

\(^7\) The idea of drawing a sequence randomly is due to Hildenbrand (1974). Anderson (1985) uses this idea to study core convergence with non-convex preferences.
Let \((N^0, V)\) be an initial game with composition \(a^{N^0} \in \Delta\), and let \((mN^0, V)\) be a game with the player set \(N^0\) replicated \(m\) times, where \(m\) is a positive integer. Then the second condition of Theorem 1 holds trivially, since a replication game has the same composition as the initial game.

**Corollary 2.** Suppose \((N^0, V)\) is a game with \(a^{N^0} \in \Delta\) and that \(v\) is differentiable at \(a^{N^0}\). Let \(\alpha \in (0, 1]\), \(\delta > 0\) be given. Then there exist \(n, \varepsilon_0 > 0\) such that

\[m \mid N^0 \mid \geq n,\]
\[\varepsilon \in [0, \varepsilon_0],\]
\[\mid S \mid \geq \alpha m \mid N^0 \mid, \text{ where } S \text{ is a coalition in } (mN^0, V), \text{ and}\]
\[U \in \mathcal{C}_e(mN^0, V) \text{ and } w \in \mathcal{C}_e(mN^0, V),\]

then \(\mid w \cdot A^S - U^S \mid < \delta \mid A^S \mid\).

**Corollary 3.** Let \(\alpha \in (0, 1]\), \(\delta > 0\). Under the hypotheses of Theorem 1, if \(U, U' \in \mathcal{C}_e(N, V)\), then \(U\) is 'close' to \(U'\) in the sense that if \(\mid S \mid \geq \alpha \mid N \mid\), then \(\mid U^S - U'^S \mid < 2\delta \mid A^S \mid\).

Corollary 4 shows that Theorem 1 can be extended to the 'types' case where a player is defined to be of type \(t\) if he possesses one unit of attribute \(t\). That is, \(A^t \in \{e^t \mid t = 1, \ldots, T\}\), where \(e^t\) is the \(t\)th unit vector in \(R^T\), i.e. \(e^t = (0, 0, \ldots, 1, \ldots, 0)\) with 1 in the \(t\)th place. In this case the superadditive payoff function \(V\) is then defined only on \(Z^+_T \setminus \{0\}\), the integral vectors in \(R^T_+ \setminus \{0\}\). We extend \(V\) to all of \(R^T_+ \setminus \{0\}\) by letting \(V^* : R^T_+ \setminus \{0\} \rightarrow R_+\) by \(V^*(A) = V(A)|A|\), where \(|A|\) is the largest element of \(Z^+_T\) less than or equal to \(A\). Since \(V^*\) is in fact a superadditive extension of \(V\), and since all coalitions in the game will be in \(Z^+_T \setminus \{0\}\), the core is the same whether we define payoffs using \(V^* \) or \(V\). We define \(\nu : \Delta \rightarrow R_+\) as before.\(^8\)

**Corollary 4.** Let \(V : Z^+_T \setminus \{0\} \rightarrow R_+\) be superadditive. Suppose that \(v\) is differentiable at \(a^{N^0} \in \Delta\) and that \(A^t \in \{e^t \mid t = 1, \ldots, T\}\). Let \(\alpha \in (0, 1]\), \(\delta > 0\) be given. Then there exist \(n, r, \varepsilon_0 > 0\) such that

\[\mid N \mid \geq n,\]
\[\| a^{N^0} - a^{N^0} \| \leq R,\]

\(^8\)The function \(\nu\) is the unique convex (hence continuous) extension of \(\tilde{\nu} : \Delta \cap Q^T \rightarrow R_+\) defined by \(\tilde{\nu}(a) = \sup \{V(ra)/r \mid r \in Z^+_T, a \in \Delta \cap Q^T\}\). We have derived the extension \(\nu\) from a particular superadditive extension \(V^*\) of \(V\), but in fact \(\nu\) is uniquely determined by \(V\).
(3) $\epsilon \in [0, \epsilon_0]$.

(4) $|S| \geq \alpha |N|$, and

(5) $U \in C(N, V)$ and $w \in C(N, V)$,

then $|w \cdot A^S - U^S| < \delta |A^S|$.

If all that is known about a player set is that there are $\tau$ different types of players, then the natural game is the one described in Corollary 4: each player has one unit of one of $\tau$ attributes. The conclusion of Corollary 4 is equal treatment in the limit: in the limit each non-trivial group of players of type $t$ receives the same per capita payoff as any other non-trivial group of players of type $t$. However, we might know more than merely that there are $\tau$ types of players. Suppose that each player's attribute vector is in $R^T_\tau$. Then if $T < \tau$, the game with types in $R^T_\tau$ is 'throwing away' information. We could define an alternative game where each player's 'type' is a point in $R^T_\tau$ instead of $R^T_\tau$. Theorem 1 would again imply equal treatment in the limit, but it also implies something stronger. The payoff to each type of player can be decomposed according to a linear function on his attributes, and the same linear function applies to all types. Furthermore, since $R^T_\tau$ has a natural metric, embedding the players' attributes in $R^T_\tau$ provides a natural way to say when players of different types are similar. And the hedonic payoff $w$ provides a natural way to say that players with similar attributes receive similar payoffs. This structure is lost in the game where we define the $\tau$ types as $\tau$ different attributes. 9

In the appendix we prove Theorem 1. Here we give the idea of the proof.

The function $\nu$ is concave, hence differentiable almost everywhere on its domain, and Theorem 1 uses the assumption that $\nu$ is differentiable at $a^F$ with the supporting hyperplane $w^F$. We show that every payoff $U$ in the $\epsilon$-core is close to the hedonic payoffs $(w^F \cdot A^i)_{i \in N}$. We give the intuition for this result below. For payoffs $w$ in the $\epsilon$-hedonic core, $(w \cdot A^i)_{i \in N}$ is in the $\epsilon$-core and therefore close to $(w^F \cdot A^i)_{i \in N}$ and to $U$. The theorem follows.

Part I. $w^F \cdot a^S - u^S < \delta$.

Suppose not. In Fig. 3 we have drawn the case that $u^S \leq w^F \cdot a^S - \delta$. If $a^S$ were close to $a^F$, we would have $u^S < \nu(a^S)$, and for a large enough player set, which by Lemma A.2 in the appendix implies that $V(A^S)/|A^S|$ is close to $\nu(a^S)$, coalition $S$ could $\epsilon$-block. Thus we only need to argue for the case that $a^S$ is bounded away from $a^F$, as in Fig. 3. If $a^N$ is close to $a^F$, then $a^S$ is also bounded away from $a^N$. Since $|A^S|$ is a non-trivial fraction of $|A^N|$, it follows that $a^{N\setminus S}$ is also bounded away from $a^F$ and $a^N$, as in Fig. 3.

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9 Subsequent to circulation of this paper, Wooders (1992) has circulated an 'equal treatment' theorem for the $\epsilon$-core for players with attributes which relies on a scale assumption. The equal-treatment payoffs are rewards to bundles of attributes (that is, players have 'types'), and not a linear function on attributes as here.
We construct a blocking coalition, \( N \setminus S_1 \). To find the coalition \( S_1 \) that will be excluded from \( N \), we divide the coalition \( N \setminus S \) into \( k \) sub-coalitions, each with composition close to \( a^{N \setminus S} \). Lemma A.4 in the appendix tells us that we can do this, and that the compositions can be as close as we like to \( a^{N \setminus S} \), provided \( |A^{N \setminus S}| \) is large. Of the \( k \) coalitions, \( S_1 \) will be one with the highest per-attribute payoff \( u^{S_1} \geq u^{N \setminus S} \).

The intuition for why \( N \setminus S_1 \) can \( \epsilon \)-block follows by supposing it holds exactly that \( a ^{S_1} = a^{N \setminus S} \). As long as \( |A^{S_1}| \) is not too large relative to \( |A^{N \setminus S}| (k \) is not too small), then \( a^{N \setminus S} \in D \), where \( D \) is shown in Fig. 3. Using Lemma 1, if \( u^{S_1} \geq u^{N \setminus S} \), then \( u^{N \setminus S} \) lies on or below the dark line in Fig. 3 in the domain \( D \), and strictly below \( v \). (This is where we use the fact that there is a unique supporting hyperplane at \( a^{F} \); as a consequence the line connecting \( u^{N \setminus S} \) with \( u^{S} \) lies somewhere below \( v \).

Since \( u^{N \setminus S} \) lies below the dark line, we choose \( D \) so that for some \( \epsilon, \phi > 0 \), \( u^{N \setminus S} \) lies below \( v(a^{N \setminus S}) - \epsilon - \phi \). By uniform convergence of \( V(ra)/r \) to \( v(a) \), a large enough coalition with composition \( a^{N \setminus S} \) can achieve a payoff greater than \( v(a) - \phi \) per unit attribute. Since the utility per unit attribute that the coalition \( N \setminus S_1 \) achieves, namely \( u^{N \setminus S_1} \), is less than \( v(a^{N \setminus S}) - \epsilon - \phi \), \( N \setminus S_1 \) can improve its utility by more than \( \epsilon \) \( |A^{N \setminus S}| \) and can therefore \( \epsilon \)-block. (The formalization of this argument in the appendix recognizes that \( a^{S_1} \) is only approximately equal to \( a^{N \setminus S} \).

**Part II.** \( u^S - w^F \cdot a^S < \delta \).

Suppose not. Then \( u^S \) is as shown in Fig. 4. Since \( u^{N} \leq v(a^{N}) \leq w^F \cdot a^{N} \), hedonic payoffs strictly larger than \( w^F \) are infeasible for the players, on average. Members of \( S \) in Fig. 4 receive significantly more payoff to their attributes than \( w^F \), on average, and therefore members of \( N \setminus S \) receive less than \( w^F \), on average. Since coalition \( S \) is a significant fraction of the player set, each member of \( N \setminus S \) provides a significant 'subsidy' to \( S \); for some \( \omega > 0 \), \( w^F \cdot a^{N \setminus S} - u^{N \setminus S} > \omega \). Since \( V > 0 \), \( \epsilon \)-core payoffs are bounded below, and so in order for \( N \setminus S \) to give a significant subsidy to \( S \), it must itself constitute a non-trivial fraction of the player set.
set. However, then we can apply Part I, using $N \setminus S$ in place of $S$, to argue that such a payoff cannot be in $C_\epsilon(N, V)$.

4. Comparative statics

The most intuitive notion in economics is that scarcity leads to high rents. Scarcity is usually interpreted to mean scarcity of a commodity, for example, economists often use partial equilibrium models in which a reduction in the supply of a commodity increases its own price. This result is harder to obtain in general equilibrium where all prices change in response a reduction in supply, but there are known conditions under which it is true (see Engl and Scotchmer, 1994). Scarcity can also mean scarcity of agents. In the context of one-to-one or many-to-many matching models, Crawford (1991) shows that a larger number of players on one side of the match will reduce their equilibrium payoffs. Scotchmer and Wooders (1988) considered a class of games with a scale property and showed that an increase in the number of agents of one type will lead to a decrease in the utility received by each player of that type in the core, and Scotchmer (1994) extended the result to proportions of players rather than simply numbers. The scale property implies that only small coalitions have blocking power.

The comparative statics below, which do not require a scale assumption, are on attributes: if we have two player sets that weight a particular attribute differently, the hedonic core will reward that attribute more in the game where it is scarce. A person with one unit of attribute $t$ can be defined as a person of type $t$, and then the result means that if one type becomes relatively more plentiful, that type will be rewarded less.

Fig. 5 provides the intuition for how the comparative static result relates to the exhaustion of blocking opportunities. It is easiest to provide the intuition if we assume, as in Fig. 5, that $V$ is homogeneous, so that if $w$ is in the hedonic core or
\( \epsilon \)-hedonic core of \((N, V)\), then \( w \cdot a^N = v(a^N) = V(A^N)/|A^N| \). Whether or not \( V \) is homogeneous, a payoff \( w \) in the hedonic core or \( \epsilon \)-hedonic core of \((N, V)\) need not correspond to a supporting hyperplane to \( v \) at \( a^N \); there might be points in the simplex where \( v \) lies above the linear function \( w \), and similarly for a payoff \( w' \) in the hedonic core or \( \epsilon \)-hedonic core of \((N', V)\). But our concept of exhaustion ensures that, if \( w \) is in the hedonic core or \( \epsilon \)-hedonic core, then it corresponds to a linear function that is close to a supporting hyperplane. This is true whether or not \( V \) is homogeneous.

In Fig. 5 the game \((N', V)\) has a higher fraction of attribute 2 than \((N, V)\) and a smaller fraction of attribute 1; we argue that, as a consequence, \( w'_1 \geq w_1 \) and \( w'_2 \leq w_2 \). This follows from the slopes of the lines that represent \( w \) and \( w' \). The directional derivative of the linear function \( A \mapsto w \cdot A \) in the direction from \((1, 0)\) to \((0, 1)\) is \((w_2 - w_1)/\sqrt{2}\), and similarly for the linear function \( A \mapsto w' \cdot A \). Provided blocking opportunities are approximately exhausted, \( w \) lies above \( v \) at \( a^N \) and \( w' \) lies above \( v \) at \( a^N \) (Lemma 2 below). Thus the slope of the line representing \( w \) must be larger than the slope of the line representing \( w' \), which implies that \( w_2 - w_1 > w'_2 - w'_1 \). We cannot have \( w \gg w' \), since that would imply that the line representing \( w \) would lie everywhere above \( w' \), and then \( w' \) could not be close to a supporting hyperplane. Similarly, we cannot have \( w' \gg w \). Therefore \( w'_1 \geq w_1 \) and \( w'_2 \leq w_2 \).

The argument suggested by Fig. 5 must be generalized in two ways. First, we must generalize it to more than two dimensions. If the second attribute is more heavily represented in \( N' \) than in \( N \), while the other attributes are proportionately diminished, then \( w_2 \geq w'_2 \), but we cannot say which other payoffs increase or decrease. Second, we must generalize to the case when \( V \) is not homogeneous. If \( V \) is not homogeneous, then we will be concerned with the \( \epsilon \)-hedonic core rather than the core, since typically the core will be empty. \( V(A^N)/|A^N| \) may lie below \( v(a^N) \), but if blocking opportunities are almost exhausted, as defined below, then \( V(A^N)/|A^N| \) will be close to \( v(a^N) \). It follows that if \( w \in C^H_\epsilon(N, V) \), then \( w \cdot a^N = V(A^N)/|A^N| \) will be only slightly smaller than \( v(a^N) \), and the argument depicted in Fig. 5 will apply in an approximate sense. Both these generaliza-
tions are captured in Theorems 2 and 3 below, which are the main results of this section.

We begin by defining the 'approximate exhaustion of blocking opportunities', which will allow us to conclude that a payoff \( w \) in the hedonic core or \( \epsilon \)-hedonic core is close to a supporting hyperplane.

As a preliminary step in defining the set of 'blocking opportunities', we first define the set of compositions that coalitions in a game might have. When the players' attributes are drawn independently from a distribution \( F \), this set is

\[
\Delta^F = \{ a \in A \mid a = A / | A| \text{ for some } A \in \text{the convex hull of the closed support}^{10} \text{ of } F \}
\]

And when the player set is a replica of an initial player set \( N^0 \) with attributes \((A^i)_{i \in N^0}\), this set is

\[
\Delta^0 = \{ a \in A \mid a = A / | A| \text{ for some } A \in \text{the convex hull of } (A^i)_{i \in N^0} \}
\]

If, for each attribute \( t \), \( F \) puts weight on the possibility that a player has a positive quantity of only that attribute, then \( \Delta^F = \Delta \). Or if for each attribute \( t \) the initial replica game \( N^0 \) has a player with only that attribute, then \( \Delta^0 = \Delta \).

For an arbitrary set of compositions \( \Delta^* \) that coalitions in a game might have (taken to be \( \Delta^0 \) or \( \Delta^F \) below), the set of 'blocking opportunities' is

\[
\Omega_0(\Delta^*) = \{ w \in R_+^T \mid w \cdot a < \nu(a) \text{ for some } a \in \Delta^* \}
\]

Such a set is drawn in Fig. 6 for the case that \( \Delta^* = \Delta \). Each \( w \) in \( \Omega_0(\Delta^*) \) corresponds to a linear function \( w \) in Fig. 5 that lies somewhere below \( \nu \) in \( \Delta^* \).

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\(^{10}\) Every neighborhood of a point in the closed support of a distribution has positive measure.
For every \( w \in \Omega_0(\Delta^*) \) (and for no \( w \) not in \( \Omega_0(\Delta^*) \)) we can imagine a game \((N, V)\) that contains a coalition \( S \) for which \( V(A^S) > w \cdot A^S \), and this is why we call \( \Omega_0(\Delta^*) \) the set of blocking opportunities. In a finite game, not all the vectors in \( \Omega_0(\Delta^*) \) can be blocked (typically), but if we enlarge the player set, say to \( N' \supset N \), then the set of vectors in \( \Omega_0(\Delta^*) \) that can be blocked in the game \((N', V)\) is larger than in the game \((N, V)\). If every \( w \in \Omega_0(\Delta^*) \) could be blocked, then a payoff \( w \) in the hedonic core would be in the boundary of \( \Omega_0(\Delta^*) \). Payoffs in the boundary of \( \Omega_0(\Delta^*) \) correspond to supporting hyperplanes to \( \nu \).

In a large game 'almost' all the vectors in \( \Omega_0(\Delta^*) \) can be blocked. We now formalize this notion. Given \( a \in \Delta \) and \( \epsilon > 0 \) we denote the ball of radius \( \epsilon \) about \( a \) in the simplex by \( B(a, \epsilon) = \{ a' \in \Delta : \| a' - a \| < \epsilon \} \). The interior of \( \Delta^* \) is \( \text{int} \Delta^* = \{ a \in \text{int} \Delta : B(a, \epsilon) \subset \Delta^* \text{ for some } \epsilon > 0 \} \) and the boundary of \( \Delta^* \) is \( \partial \Delta^* = \Delta^* \setminus \text{int} \Delta^* \). For \( \epsilon \geq 0 \), let

\[
\Omega_\epsilon(\Delta^*) = \{ w \in R^r_+ | w \cdot a < v(a) - \epsilon \text{ for some } a \in \Delta^*, d(a, \partial \Delta^*) \geq \epsilon \}.
\]

To say that a game is 'large', we use the concept of \( \epsilon \)-exhaustion of blocking opportunities. We say that a game \((N, V)\) \( \epsilon \)-exhausts blocking opportunities if there exists a coalition \( S \) such that \( V(A^S)/|A^S| > w \cdot A^S \) for each \( w \in \Omega_\epsilon(\Delta^0) \) in the replica case and for each \( w \in \Omega_\epsilon(\Delta^f) \) in the case that the players' attributes are drawn randomly. Restricting attention to randomly drawn attributes, this means that every \( w \) in \( \Omega_\epsilon(\Delta^f) \) can be blocked. It follows that payoffs in the hedonic core lie between the boundaries of \( \Omega_\epsilon(\Delta^f) \) and \( \omega_0(\Delta^f) \). Similarly, the arguments in Lemma 2 below show that payoffs in the \( \epsilon \)-hedonic core lie between the boundaries of \( \Omega_2\epsilon(\Delta^f) \) and \( \Omega_0(\Delta^f) \), as shown in Fig. 7. Such payoffs correspond to the linear functions \( w \) in Fig. 5 that are close to supporting hyperplanes. Proposition 1 below shows that a large enough game \( \epsilon \)-exhausts blocking opportu-
Proposition 1. Suppose the attributes of the players in a game \((N, V)\) are drawn independently according to a distribution \(F\). Given \(\epsilon > 0\) and \(\pi \in [0, 1]\) there exists \(r^*\) such that if \(|N| \geq r^*\), then \((N, V)\) \(\epsilon\)-exhausts blocking opportunities with probability at least \(\pi\).

Proof. Let \(\epsilon > 0\) be given. We show first that \(\Omega_\epsilon(\Delta^F)\) is bounded, i.e. there exists \(M\) such that \(\|w\| \leq M\) for all \(w \in \Omega_\epsilon(\Delta^F)\). By Lemma A.1 in the appendix, \(v\) is concave and hence continuous on int \(\Delta\), and so we let \(\tilde{v}\) be an upper bound for \(v\) on \(\{a \in \Delta \mid d(a, \partial \Delta) \geq \epsilon\}\). If \(\Omega_\epsilon(\Delta^F)\) were unbounded, then we could find a sequence \(w^n \in \Omega_\epsilon(\Delta^F)\) such that \(\|w^n\| \to +\infty\), and since \(w^n \geq 0\) this would imply that \(w^n \cdot a^n \to +\infty\) for any sequence \(\{a^n\}\) in \(\Delta\) with \(d(a^n, \partial \Delta) \geq \epsilon\) and \(w^n \cdot a^n < v(a^n) - \epsilon \leq \tilde{v} - \epsilon\), which leads to a contradiction. Thus there exists \(M\) such that \(\|w\| \leq M\) for all \(w \in \Omega_\epsilon(\Delta^F)\).

By Lemma A.2 in the appendix, \(V(rs)/r\) converges uniformly to \(v\) on compact subsets of int \(\Delta\), so we can find \(r^*\) such that if \(|N| \geq r^*\), then for any \(a \in \Delta^F\) with \(d(a, \partial \Delta) \geq \epsilon\) with probability at least \(\pi\) there exists a coalition \(S\) such that

\[
\frac{A^S}{|A^S|} \in B\left(a, \frac{\epsilon}{2}\right) \quad \text{and} \quad \frac{V(A^S)}{|A^S|} \geq v\left(\frac{A^S}{|A^S|}\right) - \epsilon \geq \frac{v(a^n) - \epsilon}{2}.
\]

We fix a player set such that \(|N| \geq r^*\). We show that the game \((N, V)\) \(\epsilon\)-exhausts blocking opportunities. That is, given \(w \in \Omega_\epsilon(\Delta^F)\) we show that there exists a coalition that blocks \(w\).

By concavity, \(v\) is in fact Lipschitz on \(\{a \in \Delta \mid d(a, \partial \Delta) \geq \epsilon/2\}\), so let \(m \geq 1\) be a Lipschitz constant. Given \(w \in \Omega_\epsilon(\Delta^F)\), we define \(\nu: \Delta \to R\) by \(\nu(a) = v(a) - w \cdot a\). Then \(M + m\) is a Lipschitz constant for \(\nu\) on \(\{a \in \Delta \mid d(a, \partial \Delta) \geq \epsilon/2\}\).

By the definition of \(\Omega_\epsilon(\Delta^F)\), we can find \(a_u \in \Delta^F\) such that \(\nu(a_u, \partial \Delta) > \epsilon\) and \(\nu(a_u) > \epsilon\). Then \(B(a_u, \epsilon/[2(M + m)]) \subset \{a \in \Delta \mid d(a, \partial \Delta) \geq \epsilon/2\}\) and so for \(a \in B(a_u, \epsilon/[2(M + m)])\) we have \(|\nu(a) - \nu(a_u)| \leq (M + m)(\epsilon/[2(M + m)]) = \epsilon/2\). Since \(\nu(a_u) > \epsilon\), we have \(\nu(a) > \epsilon/2\) for all \(a \in B(a_u, \epsilon/[2(M + m)])\). However, by (1) with probability at least \(\pi\), there exists a coalition \(S\) such that \(A^S/|A^S| \in B(a_u, \epsilon/2)\) and \(V(A^S)/|A^S| \geq v(A^S/|A^S|) - \epsilon/2\). But \(\nu(A^S/|A^S|) - w \cdot A^S = v(A^S/|A^S|) > \epsilon/2\) and so \(V(A^S)/|A^S| > w \cdot A^S/|A^S|\), which implies that \(V(A^S) > w \cdot A^S\), i.e. \(S\) can block \(w\). Therefore, \((N, V)\) \(\epsilon\)-exhausts blocking opportunities. \(\Box\)

Lemma 2 and Proposition 2 below are stated for players with randomly drawn attributes, but the same proof applies for replica games. Lemma 2 is used to prove asymptotic monotonicity, Proposition 2, and comparative statics, Theorems 2 and
3. By ε-exhaustion of blocking opportunities, we conclude in Lemma 2 that $w \cdot a^{N'}$ is (almost) above $v(a^{N'})$ and $w' \cdot a^{N'}$, and that $w' \cdot a^N$ is (almost) above $v(a^N)$ and $w \cdot a^N$, as in Fig. 5.

Lemma 2. Suppose the attributes of the players in a game $(N, V)$ are drawn independently according to some distribution $F$. Given $\psi > 0$ there exists $\epsilon > 0$ such that if

1. $(N, V)$ and $(N', V)$ $\epsilon$-exhaust blocking opportunities,
2. $a^N, a^{N'} \in \text{int}(\Delta^F \cap \Delta^{F'})$, and
3. $w \in C^H_e(N, V)$ and $w' \in C^H_e(N', V)$,

then $w \cdot a^{N'} \geq w' \cdot a^{N'} - \psi$ and $w' \cdot a^N \geq w \cdot a^N - \psi$.

Proof. Let $2\epsilon < \min\{\psi, d(a^N, \partial(\Delta^F \cap \Delta^{F'})), d(a^{N'}, \partial(\Delta^F \cap \Delta^{F'}))\}$. We show the first inequality. If it were not true, then $w \cdot a^{N'} < w' \cdot a^{N'} - \psi < v(a^{N'}) - \psi < v(a^{N'}) - 2\epsilon$, which implies that $w \in \Omega_{2\epsilon}(\Delta^F \cap \Delta^{F'})$. Let $1 = (1, 1, \ldots, 1) \in \mathbb{R}^T$. Then since $w \in \Omega_{2\epsilon}(\Delta^F \cap \Delta^{F'})$, we have $(w + \epsilon 1) \in \Omega_{\epsilon}(\Delta^F \cap \Delta^{F'}) \subset \Omega_{\epsilon}(\Delta^F)$, and therefore there exists a coalition $S$ such that $V(A^S) / |A^S| > (w + \epsilon 1) \cdot a^S$. Thus $w \not\in C^H_e(N, V)$, a contradiction. □

The next proposition follows immediately.

Proposition 2 (asymptotic monotonicity). Suppose that the attributes of the players in a game $(N, V)$ are drawn independently according to a distribution $F$. Given $\psi > 0$ there exists $\epsilon > 0$ such that if

1. $(N, V)$ and $(N', V)$ $\epsilon$-exhaust blocking opportunities,
2. $a^N, a^{N'} \in \text{int}(\Delta^F \cap \Delta^{F'})$, and
3. $w \in C^H_e(N, V)$ and $w' \in C^H_e(N', V)$,

then $(w' - w) \cdot (a^{N'} - a^N) \leq 2\psi$.

Asymptotic monotonicity seems to be of limited interest because it does not imply the comparative statics contained in Theorems 2 and 3, as shown by the following example. \footnote{Our argument for comparative statics relies on Lemma 2, as does our argument for asymptotic monotonicity. We notice that the argument of Wooders (1992), who showed asymptotic monotonicity using a scale assumption similar to that of Scotchmer and Wooders (1988), does not contain an analog to Lemma 2, and does not imply comparative statics on proportions.}

Example 2. Suppose $a^N = (0.4, 0.3, 0.3)$, $a^{N'} = (0.5, 0.25, 0.25)$, $w = (2, 2, 2)$ and $w' = (3, 0, 7)$. These compositions and payoffs satisfy monotonicity on proportions since $(w' - w) \cdot (a^{N'} - a^N) = (1, -2, 5) \cdot (0.1, -0.05, -0.05) =$
-0.05 < 0, but even though $a^N$ and $a''^N$ satisfy the hypotheses of Proposition 2, $w$ and $w'$ do not satisfy the comparative statics conclusion that $w'_j \leq w_j$. □

We now prove the comparative statics results assuming that the players in a game $(N', V)$ have relatively more of the $j$th attribute than the players in a game $(N, V)$ and the relative amounts of the other attributes are the same in both games. Mathematically, this hypothesis is that $a''^N = ka^N + k\beta a''_j e^j$ for some $k, \beta > 0$.

Theorem 2 (comparative statics). Suppose $T > 1$ and that the player sets $N$ and $N'$ are, respectively, replicas of $N^0$ and $N'^0$. Given $\gamma > 0$, there exists $\epsilon > 0$ such that if

1. $(N, V)$ and $(N', V)$ $\epsilon$-exhaust blocking opportunities,
2. $a''^N, a''^N \in \text{int}(\Delta^0 \cap \Delta^0')$,
3. $w \in C^H_e(N, V)$ and $w' \in C^H_e(N', V)$, and
4. $a''^N = ka^N + k\beta a''_j e^j$ for some $j = 1, 2, \ldots, T$ and $k, \beta > 0$,

then $w'_j < w_j + \gamma$.

Proof. Given $\gamma > 0$ we choose $\theta, \psi > 0$ such that $\theta < d(a^N, \partial \Delta)$ and $\psi < [k\beta \gamma/(k + 1)]\theta((T-1)/T$. For this $\psi$ there exists an $\epsilon > 0$ such that we can use the inequalities of Lemma 2. Multiplying the second inequality in Lemma 2 by $k$ and subtracting it from the first inequality we get $(w' - w) \cdot e^j \leq (k + 1)\psi/k\beta a^N_j$. It can be verified (Engl, 1993) that $d(a^N, \partial \Delta) \geq \theta \Rightarrow a''_j \geq \theta((T-1)/T$. Hence $\psi < [k\beta \gamma/(k + 1)]\theta((T-1)/T \leq [k\beta \gamma a''_j/(k + 1)]$ and therefore $w'_j - w_j \leq (k + 1)\psi/k\beta a^N_j < \gamma$. □

Theorem 2 applies to replica games. For a player set with attributes drawn independently according to $F$ or $F'$, we cannot make hypotheses directly on $a^N$ and $a''^N$, since those are random. Rather we must make hypotheses on $a^F$ and $a''^F$, recognizing that $a^N$ will be close to $a^F$ and $a''^N$ will be close to $a''^F$ in large games. Engl (1993) and Engl and Scotchmer (1992) show how the argument can be extended to randomly drawn games. Theorem 3 records the comparative statics for that case.

Theorem 3 (comparative statics). Suppose that the attributes of the players in the games $(N, V)$ and $(N', V)$ are drawn independently according to the distributions $F$ and $F'$, respectively. Given $\gamma > 0$, there exist $\epsilon, r > 0$ such that if

1. $(N, V)$ and $(N', V)$ $\epsilon$-exhaust blocking opportunities,
2. $a^F, a''^F \in \text{int}(\Delta^F \cap \Delta^{F'})$,
3. $w \in C^H_e(N, V)$ and $w' \in C^H_e(N', V)$,
4. $a''^F = ka^F + k\beta a''_j e^j$ for some $j = 1, 2, \ldots, T$ and $k, \beta > 0$, and
5. $\|a^F - a^N\|, \|a''^F - a''^N\| < r$,

then $w'_j < w_j + \gamma$. 
For 'types' games as described in Corollary 4 above, Theorem 3 is a generalization of the asymptotic version of the Scotchmer and Wooders (1988) monotonicity result. They used a 'scale assumption' to show (translated into the framework developed here) that $(w' - w) \cdot (A^N - A^N) \leq 0$, where $w \in C_0^H(N, V)$ and $w' \in C_0^H(N', V)$. For 'types' games where $A^t_i \in \{e^t_i | t = 1, \ldots, T\}$, this inequality shows that if the number of players of one type is increased, the equal-treatment payoff to that type decreases (or, more accurately, cannot increase). However, the inequality cannot be interpreted to mean that if the proportion of one type of player increases (and, for example, all the other types of players also become more abundant), then the payoff to that type decreases. In addition, games with homogeneous $V$ such as those derived from exchange economies do not satisfy their scale assumption. Theorems 2 and 3 above cover both these situations.

5. Conclusion: Convergence of the core to competitive outcomes

We can interpret the hedonic core payoffs as competitive prices, and therefore our approximation theorem can be interpreted as a core convergence theorem. We develop two examples to show how this is so. For simplicity, in this section we use the term 'exhaustion of blocking opportunities' to mean 0-exhaustion with respect to the set $\Omega_0(\Delta)$, that is, any $w \in \Omega_0(\Delta)$ can be blocked.

Example 3: Coalition production. Coalition production is probably the most immediate example of how hedonic pricing in the core can be interpreted as hedonic pricing in markets. The characteristic function $V$ can be interpreted directly as a production technology, $V(A)$ being the value of the output produced by a firm using workers whose total attributes are given by $A$. If $w$ is a vector of wage rates, one for each attribute, then the wage of a worker with attributes $A^t_i$ is $w \cdot A^t_i$. Competitive prices must satisfy $V(A) - w \cdot A \leq 0$ for all $A \in R_+^N \setminus \{0\}$ (no firm could hire a set of workers that would earn positive profit), and $V(A^N) - w \cdot A^N = 0$ (the equilibrium wage function $w$ provides zero profit). But when blocking opportunities are exhausted, these two conditions also describe a hedonic core payoff, hence competitive equilibrium and the hedonic core coincide. Furthermore, the approximation theorem can be interpreted as saying that the approximate core converges to the competitive payoffs as the game becomes large.

In a competitive market one would expect wages to be anonymous in the sense that wages reflect productivity, and not the other characteristics of the worker. This is so. If an attribute is irrelevant to productivity in the sense that $\partial V(A^N)/\partial A^t_i = 0$, then (since $w$ defines a supporting hyperplane to $V$), $w_j = 0$. Workers will earn different wages even though they face a common hedonic wage function, but only because they have different productive attributes.
The comparative static result means that if, for example, the proportional representation of management skills in the population grows, then the hedonic price for management skills will fall. And thus a worker with good management skills will be rewarded less for those skills.

*Example 4: Clubs.* We assume that players receive utility by interacting, and that their utility depends on their own and other players' personal characteristics, such as generosity, level of education, and inclination to smoke. We will assume that the utility of agent $i$ can be represented as $A^i \cdot \phi(A^i)$, where $S$ is a club that agent $i$ belongs to, and $\phi: R_+^T \setminus \{0\} \to R_+^T$. The function $\phi$ represents the externalities received by the players. We assume that $\phi = (\phi^1, \ldots, \phi^T)$ is differentiable and that $\phi^1 \equiv 1$, so that $A^i$ is agent $i$'s endowment of a quasilinear good. If the $j$th attribute produces no externalities, then $\partial \phi^k / \partial A^j = 0$ for all $k$. If the $j$th attribute only produces externalities, and does not aﬀect its owner's utility directly, then $\phi^j = 0$. Some attributes, such as smoking, aﬀect both one's own utility and others' utility.

We let $E(A)_n = A \cdot (\partial \phi / \partial A^X(A))$. $E(A^S)_n$ gives the marginal externality imposed on a club $S$ with attributes $A^S$ by an additional unit of the $n$th attribute. The total utility of $S$ is $A^S \cdot \phi(A^S)$. The additional unit of the $n$th attribute affects the utility of $S$ directly through a change in $A^S$ as well as indirectly through a change in $\phi(A^S)$. $E(A^S)_n$ measures this indirect effect. We let $E(A) = (E(A)_1, \ldots, E(A)_n)$. We shall refer to the club economy as $(N, \phi)$, and to induce a corresponding game $(N, V)$ we define the superadditive function $V$ by

$$V(A) = \sup \left\{ \frac{\sum_{S \in C} A^S \cdot \phi(A^S)}{\sum_{S \in C} A^S} = A, \text{ C finite} \right\}.$$  

Because $\phi$ is homogeneous of degree 0, $V$ is homogeneous of degree 1.

Our objective is to argue that a hedonic core payoff can be interpreted as a competitive price system, and that our approximation theorem can be interpreted as saying that core payoffs converge to competitive payoffs.

We now consider how to decentralize the core as a competitive equilibrium using hedonic admission prices. A *hedonic admission price* is a function $p: R_+^T \setminus \{0\} \to R^T$, where $p(A) \cdot A^i$ is the amount that $i$ must pay for admission to a club with total attributes $A$. A *competitive equilibrium* for $(N, \phi)$ is an ordered pair $(C, p)$, where $C$ is a partition of $N$ and $p$ is a hedonic admission price, such that

(i) no consumer could increase utility by buying admission to any other coalition:

$$[\phi(A^S) - p(A^S)] \cdot A^i \geq [\phi(A) - p(A)] \cdot A^i \text{ for all } i \in S \in C,$$

$$A \in R_+^T \setminus \{0\};$$

(ii) clubs (coalitions) in equilibrium make zero profit, and no conceivable club
could provide positive profit:

\[ p(A) \cdot A \leq p(A^S) \cdot A^S = 0 \text{ for all } S \in C, \ A \in R^T_+ \setminus \{0\}. \]

The *competitive payoff* associated with \((C, p)\) is \(W = (W^i)_{i \in N}\), where \(W^i = [\phi(A^S) - p(A^S)] \cdot A^i, i \in S \subseteq C\). As with core payoffs, we let \(W^S\) represent the sum of competitive payoffs in a coalition \(S\), i.e. \(W^S = \sum_{i \in S} W^i\).

**Proposition 3** (Hedonic-core payoffs are competitive payoffs). Let \((N, V)\) be the game derived from a club economy \((N, \phi)\). Suppose that

1. \((N, V)\) exhausts blocking opportunities,
2. \(w \in C_0^H(N, V)\), and
3. \(w \cdot A^N = V(A^N) = \sum_{S \subseteq C} A^S \cdot \phi(A^S) = \sum_{S \subseteq C} V(A^S)\) for some partition \(C\) of \(N\).

Then

4. there exists a competitive equilibrium \((C, p)\) and the competitive payoff is \((w \cdot A^i)_{i \in N}\), and
5. (Externality pricing) \(p(A^S) = -E(A^S)\), for all \(S \in C\) such that \(A^S \in \text{int} R^T_+\), provided \(V\) is differentiable there.

**Proof.** Let \(p : R^T_+ \setminus \{0\} \rightarrow R^T\) by \(p(A) = \phi(A) - w\). Since \(w \in C_0^H(N, V)\), by exhaustion and the definition of \(V\) we have

\[ w \cdot A \geq V(A) \geq A \cdot \phi(A) \quad \text{for all } A \in R^T_+ \setminus \{0\}. \]  

Since \(\sum_{S \subseteq C} w \cdot A^S = w \cdot A^N = \sum_{S \subseteq C} A^S \cdot \phi(A^S)\), we have

\[ w \cdot A^S = V(A^S) = A^S \cdot \phi(A^S) \quad \text{for all } S \subseteq C. \]

Then \(p(A) \cdot A = (\phi(A) - w) \cdot A \leq 0\) for all \(A \in R^T_+ \setminus \{0\}\) with equality for \(A = A^S, S \subseteq C\), so the second condition in the definition of competitive equilibrium holds. The first condition holds because \(\phi(A) - p(A) = w\) for all \(A \in R^T_+ \setminus \{0\}\). Thus \((C, p)\) is a competitive equilibrium and the competitive payoff is \((w \cdot A^i)_{i \in N}\).

If \(V\) is differentiable at \(A^S \in \text{int} R^T_+\), then by (2) and (3) \(\nabla V(A^S) = w = \phi(A^S) + E(A^S)\). Therefore, \(p(A^S) = \phi(A^S) - w = -E(A^S)\).

From the proof of Proposition 3 we see that \(p(A) = \phi(A) - w = \phi(A) - \phi(A^S) - E(A^S)\) for \(S \subseteq C\). If the \(n\)th attribute produces no externalities, then \(\partial^k \phi / \partial A^n = 0\) for all \(k\) and so \(E(A^S)_n = 0\), and thus \(p(A)_n = \phi^n(A) - \phi^n(A^S)\), for all \(S \subseteq C\). If the \(n\)th attribute produces externalities, but has no direct effect on utility, then \(\phi^n = 0\) and so \(p(A)_n = -E(A^S)_n\) for all \(S \subseteq C\). Furthermore, \(p(A) \cdot A^i = A^i \cdot \phi(A) - A^i \cdot \phi(A^S) - A^i \cdot E(A^S)\), so the amount paid by agent \(i\) to join a club with total attributes \(A\) rather than his equilibrium club \(S\) is the
difference in utility he obtains plus the reduced externalities that the members of S would receive from losing his participation. In equilibrium the amount paid to his club is the externalities that his attributes impose on the club, which may be positive or negative.

The definition of V, together with hypothesis 3, restricts the applicability of Proposition 3 to club economies where superadditivity is justified: combining attributes enhances utility. For a broader discussion of externality pricing in club economies, see Scotchmer (1995).

Let \((N^0, \phi)\) be an initial club economy and let \((mN^0, \phi)\) be the economy with the player set \(N^0\) replicated \(m\) times, where \(m\) is a positive integer. From Theorem 1 and the proof of Proposition 3 we get the following core convergence result for replication economies. We note that the hypotheses on \(w\) will hold if in particular \(\phi\) is homogeneous of degree 0 and superadditive.

**Proposition 4 (Convergence of the core to a competitive payoff).** Let \((N^0, V)\) be the game derived from the club economy \((N^0, \phi)\) and suppose that \(v\) is differentiable at \(a_{N^0} \in \text{int} A\), there exists \(w \in \mathbb{R}^T\) such that \(w \cdot A \geq V(A)\) for all \(A \in \mathbb{R}^T_+[0]\) and \(w \cdot A^{N^0} = \sum_{s \in C} A^s \cdot \phi(A^s) = \sum_{s \in C} V(A^s)\) for some partition \(C\) of \(N^0\). Let \(\alpha \in (0, 1]\), \(\delta > 0\) be given. Then there exist \(n, \varepsilon_0 > 0\) such that if

1. \(m |N^0| \geq n\),
2. \(\varepsilon \in [0, \varepsilon_0]\),
3. \(|S| \geq \alpha m |N^0|\), where \(S\) is a coalition in \(mN^0\), and
4. \(U \in C_\phi(mN^0, V)\),

then there exists a competitive payoff \(W\) for \((mN^0, \phi)\) such that \(|W^S - U^S| < \delta |A^S|\).

**Proof.** By exhaustion \(w \in C_\phi^H (mN^0, V)\), so if \(W^S = w \cdot A^S\), then by Theorem 1 \(|W^S - U^S| < \delta |A^S|\). And by Proposition 3 \((w \cdot A^s)_{s \in N^0}\) is a competitive payoff in \((N^0, \phi)\) and thus \(W = (w \cdot A^s)_{s \in mN^0}\) is a competitive payoff in \((mN^0, \phi)\).

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Appendix

More detailed versions of some of the proofs given here can be found in Engl (1993).

Lemma A.1. Let $V: \mathbb{R}_+^T \setminus \{0\} \to \mathbb{R}$ be superadditive and let $v(a) = \sup_{r > 0} V(ra)/r$ for $a \in \Delta$, then $v$ is concave.\footnote{This is part of the folklore of the subject. We include it here for completeness. The particular proof given here is similar to the one in Scotchmer and Wooders (1988).} Furthermore, either $v \equiv +\infty$ on $\text{int} \Delta$ or $v < +\infty$ on $\Delta$.

Proof. Let $a^1, a^2 \in \Delta$ be given and suppose $a = \lambda_1 a^1 + \lambda_2 a^2$ for some $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$. Let $\varepsilon > 0$ be given. Since $v(a) = \sup_{r > 0} V(ra)/r$, we can choose $n_1, n_2 \in \mathbb{N}$ such that

$$\frac{V((n_i \lambda_i) a^i)}{n_i \lambda_i} > v(a^i) - \varepsilon \quad \text{for } i = 1, 2.$$ 

Let $n = n_1 n_2$, then

$$na = n(\lambda_1 a^1 + \lambda_2 a^2) = n_1(n_1 \lambda_1) a^1 + n_2(n_2 \lambda_2) a^2,$$

so by superadditivity,

$$V(na) \geq n_1 V((n_1 \lambda_1) a^1) + n_2 V((n_2 \lambda_2) a^2).$$

Thus

$$v(a) \geq \frac{V(na)}{n} \geq \frac{1}{n_1} V((n_1 \lambda_1) a^1) + \frac{1}{n_2} V((n_2 \lambda_2) a^2)$$

$$= \lambda_1 \frac{V((n_1 \lambda_1) a^1)}{n_1 \lambda_1} + \lambda_2 \frac{V((n_2 \lambda_2) a^2)}{n_2 \lambda_2}$$

$$> \lambda_1 (v(a^1) - \varepsilon) + \lambda_2 (v(a^2) - \varepsilon) = \lambda_1 v(a^1) + \lambda_2 v(a^2) - \varepsilon.$$

But $\varepsilon$ was arbitrary, so $v(a) \geq \lambda_1 v(a^1) + \lambda_2 v(a^2)$.

The same argument can be used to establish the second statement of the lemma.

$\Box$
Lemma A.2. Let $V: \mathbb{R}^+_0 \setminus \{0\} \to \mathbb{R}$ be superadditive and let $v(a) = \sup_{r > 0} V(ra)/r$ for $a \in \Delta$, then $\lim_{r \to 0} V(ra)/r = v(a)$ and the convergence is uniform on compact subsets of $\text{int} \Delta$.

Proof. First we assume that $v$ is finite on $\Delta$. Let $\epsilon > 0$ and $a \in \Delta$ be given. We choose $r_n \to \infty$ such that $\lim_{n \to \infty} V(r_n a)/r_n = \lim \inf_{r \to \infty} V(ra)/r$. Since $v(a) = \sup_{r > 0} V(ra)/r$ we can choose $r^*$ such that

$$v(a) - V(r^* a)/r^* \leq \epsilon.$$

For $n$ large, $r_n \geq r^*$, so we choose $k_n \in \mathbb{N}$ such that

$$k_n r^* \leq r_n < (k_n + 1)r^*.$$

By superadditivity, $V(k_n r^* a) \geq k_n V(r^* a)$ so

$$V(k_n r^* a) \geq \frac{V(k_n r^*)}{k_n r^*} \geq \frac{V(r^*)}{r^*}.$$

Since $k_n r^* \leq r_n < (k_n + 1)r^*$, we have $1 \leq r_n/k_n r^* < 1 + 1/k_n$; and $k_n \to \infty$ as $r_n \to \infty$. Therefore

$$\frac{k_n r^*}{r_n} \to 1 \text{ as } n \to \infty.$$

Being superadditive and non-negative, $V$ is non-decreasing, so $V(r_n a) \geq V(k_n r^* a)$ and

$$\frac{V(r_n a)}{r_n} \geq \frac{V(k_n r^* a)}{k_n r^*} \geq \frac{V(r^* a)}{r^*} \geq (v(a) - \epsilon) \frac{k_n r^*}{r_n}.$$

Therefore $\lim_{n \to \infty} V(r_n a)/r_n \geq v(a) - \epsilon$ and, since $\epsilon$ was arbitrary,

$$\lim \inf_{r \to \infty} \frac{V(ra)}{r} = \lim_{n \to \infty} \frac{V(r_n a)}{r_n} \geq v(a) = \lim \sup_{r \to \infty} \frac{V(ra)}{r}.$$

Thus $\lim_{r \to \infty} V(ra)/r = v(a)$.

Let $K$ be a compact subset of $\text{int} \Delta$ and let $d_0 = \min_{a \in K} d(a, \partial \Delta)$, then $d_0 > 0$. Let $\Delta_0 = \{a \in \Delta | d(a, \partial \Delta) \geq d_0\}$, then $\Delta_0$ is a simplex. Being concave, $v$ is uniformly continuous on $\Delta_0$ and so given $\epsilon > 0$, $\Delta_0$ can be triangulated in such a way that $|v(a) - v(a')| \leq \epsilon$ for $a, a'$ in the same subsimplex of the triangulation. Let $\bar{v} = \max_{a \in K} v(a)$ and choose $\delta > 0$ such that $\delta T(\bar{v} - 2 \epsilon) \leq \epsilon$, then for $a \in K$, $\delta T(v(a) - 2 \epsilon) \leq \epsilon$, which implies that $-\delta T(v(a) - 2 \epsilon) \geq -\epsilon$ and so

$$(1 - \delta T)(v(a) - 2 \epsilon) \geq v(a) - 3 \epsilon \text{ for } a \in K.$$

Since $\lim_{r \to \infty} V(ra)/r = v(a)$ for $a \in \Delta$, we can choose $n_0 \in \mathbb{N}$ such that

$$r \geq n_0 \Rightarrow \frac{V(r \delta a)}{r \delta} \geq v(a) - \epsilon$$

for every vertex $a$ in the triangulation.
For \( r \geq n_2^2 \), let \( n_r \in \mathbb{N} \) such that \( n_2^2 \leq r < (n_r + 1)^2 \), then \( n_r \geq n_0 \). Furthermore, \( 1 \leq r/n_2^2 < (1 + 1/n_r)^2 \), so \( n_r^2/r < 1 \) and \( n_r^2/r \to 1 \) as \( r \to \infty \). We choose \( r_0 \geq n_0^2 \) such that \( r \geq r_0 \Rightarrow (1 - n_r^2/r)(v - 3\epsilon) \leq \epsilon \). Then for \( r \geq r_0 \) and \( a \in K \) we have \((1 - n_r^2/r)(v(a) - 3\epsilon) \leq \epsilon\), which implies that \( v(a) - 3\epsilon \leq n_r^2(v(a) - 3\epsilon)/r + \epsilon \). Thus
\[
 r \geq r_0 \Rightarrow \frac{n_r^2}{r} (v(a) - 3\epsilon) \geq v(a) - 4\epsilon \quad \text{for } a \in K.
\]

Let \( a_1, \ldots, a_r \) be the vertices of some subsimplex in the triangulation and let \( a \) be a point in this subsimplex. Then \( a = \sum_{i=1}^T \lambda_i a_i \), for some \( \lambda_i \geq 0 \) with \( \sum_{i=1}^T \lambda_i = 1 \). Let \( I = \{ i \mid \lambda_i \geq \delta \} \). If \( r \geq r_0 \), then \( n_r^2 a = \sum_{i=1}^T n_r(n_r \lambda_i a_i) \), which implies that \( V(ra) \geq V(n_r^2 a) \geq \sum_{i=1}^T n_r V(n_r \lambda_i a_i) \), so
\[
 v(a) \geq \frac{V(ra)}{r} \geq \frac{V(n_r^2 a)}{n_r^2} \frac{n_r^2}{r} \geq \frac{n_r^2}{r} \sum_{i=1}^T \frac{1}{n_r} V(n_r \lambda_i a_i)
\]
\[
 \geq \frac{n_r^2}{r} \sum_{i \in I} \lambda_i \frac{V(n_r \lambda_i a_i)}{n_r \lambda_i} \geq \frac{n_r^2}{r} \sum_{i \in I} \lambda_i (v(a_i) - \epsilon)
\]
\[
 \geq \frac{n_r^2}{r} \sum_{i \in I} \lambda_i (v(a) - 2\epsilon) \geq \frac{n_r^2}{r} (1 - 7\delta) (v(a) - 2\epsilon)
\]
\[
 \geq \frac{n_r^2}{r} (v(a) - 3\epsilon) \geq v(a) - 4\epsilon.
\]
Thus \( V(ra)/r \) converges uniformly to \( v(a) \) on \( \Delta_0 \) and hence on \( K \).

With slight modifications the same argument also works in the case that \( v \equiv +\infty \) on \( \text{int} \Delta \). \( \square \)

**Lemma A.3.** If \( A^n \in \text{int} \mathbb{R}_+^T \), \( v(a^n) < +\infty \) and \( \epsilon \geq v(a^n) - V(A^n)/|A^n| \), then \( C^\epsilon(N, V) \neq \emptyset \).

**Proof.** Let \( \hat{V} : \mathbb{R}_+^T \setminus \{0\} \to \mathbb{R} \) by \( \hat{V}(A) = |A| v(A/|A|) \), and let \( w_0 \) be a supporting hyperplane to \( \hat{V} \) at \( A^n \), i.e. \( w_0 \in \mathbb{R}^T \) and
\[
(1) \quad w_0 \cdot A^n = \hat{V}(A^n),
\]
\[
(2) \quad w_0 \cdot A \geq \hat{V}(A) \quad \text{for every } A \in \mathbb{R}_+^T \setminus \{0\}.
\]

By the second condition,
\[
w_0 \cdot a \geq \hat{V}(a) = v(a) \quad \text{for every } a \in \Delta.
\]

Let
\[
w = w_0 - \frac{\hat{V}(A^n) - V(A^n)}{|A^n|} 1,
\]

where \( 1 = (1, 1, \ldots, 1) \in \mathbb{R}^T \).
Claim. \( w \in C^h_e(N, V) \).

1. (Feasibility)
\[
w \cdot A^N = w_0 \cdot A^N - \left[ \hat{V}(A^N) - V(A^N) \right] = V(A^N).
\]

2. (No coalition can improve by more than \( \epsilon \) per ‘capita’.)

Suppose there exists a coalition \( S \) such that
\[
V(A^S) > w \cdot A^S + \epsilon |A^S| = w_0 \cdot A^S - \frac{|A^S|}{|A^N|} \left[ \hat{V}(A^N) - V(A^N) \right] + \epsilon |A^S|.
\]

This implies
\[
v(a^S) \geq \frac{V(A^S)}{|A^S|} > w_0 \cdot a^S - \frac{1}{|A^N|} \left[ \hat{V}(A^N) - V(A^N) \right] + \epsilon
\]
\[
\geq v(a^S) - \frac{1}{|A^N|} \left[ \hat{V}(A^N) - V(A^N) \right] + \epsilon
\]
\[
= v(a^S) - \frac{V(A^N)}{|A^N|} + \epsilon.
\]

But this implies that
\[
v(a^N) - \frac{V(A^N)}{|A^N|} > \epsilon,
\]

which contradicts the hypothesis on \( \epsilon \). \( \Box \)

Proposition A.1. Let \( K \) be a compact subset of \( \text{int} \Delta \). Given \( \epsilon > 0 \), there exists \( n_0 \) such that if \( |A^N| > n_0 \) and \( a^N \in K \), then \( C^h_e(N, V) \neq \emptyset \).

Proof. This is an immediate consequence of the previous lemmas. \( \Box \)

Lemma A.4. Let \{\( A^i \)\} be a sequence in \( \mathbb{R}^+_\text{t} \), with \( 0 < A \leq |A^i| \leq \tilde{A} \) for all \( i \). Let \( k \in \mathbb{N} \) and \( r > 0 \) be given. There exists an \( n \in \mathbb{N} \) such that if \( n_1, \ldots, n_k \in \mathbb{N} \) with \( n_1, \ldots, n_k \geq n \) and \( S \subset \mathbb{N} \) with \( |S| = \sum_i n_i \), then there exist \( S_1, \ldots, S_k \subset S \) such that
\[\bigcup_i S_i = S, |S_i| = n_i \text{ and } \| a^{S_i} - a^S \| < r.\]

Proof. The proof is by induction on \( k \).

\( k = 2 \). Suppose not, then there exists \( r > 0 \) such that for every \( n \in \mathbb{N} \) there exist \( n_1, n_2 \geq n \) and \( S \subset \mathbb{N} \) with \( |S| = n_1 + n_2 \) such that there do not exist \( S_1, S_2 \subset S \) with \( S_1 \cup S_2 = S, |S_1| = n_i \) and \( \| a^{S_i} - a^S \| < r \).

We give the idea of the proof with reference to Fig. A.1. Choose \( S_1, S_2 \subset S \) with \( S_1 \cup S_2 = S, |S_i| = n_i \) and with \( \| a^{S_i} - a^S \| \) minimal. Consider the hyperplane through \( a^{S_i} \) perpendicular to \( p = a^{S_i} - a^{S_2} \). Since \( a^{S_i} \) is a convex combina-
tion of the points $a^i, i \in S_1$, there must be an $i_1 \in S_1$ such that $a^{i_1} \in H_1$, the closed half space to the left of $a^{S_1}$. Similarly, there is an $i_2 \in S_2$ such that $a^{i_2} \in H_2$, the closed half space to the right of $a^{S_2}$. We trade $i_1$ for $i_2$ to get coalitions $S'_1$ and $S'_2$ with compositions $a^{S'_1}$ and $a^{S'_2}$. The vector $a^{S'_1} - a^{S'_2}$ points away from $H_1$ and away from $a^{S_1}$ as shown. It cannot be parallel to the boundary of $H_1$ because in creating $S'_1$ from $S_1$ we add a player whose composition is in $H_2$ (and in the simplex) and subtract one whose composition is in $H_1$. And the distance between $a^{S'_1}$ and $a^{S_1}$ cannot be too great if the coalitions $S_1$ and $S'_1$ have many other players. Reasoning similarly for $S'_2$, we conclude that $a^{S'_1}$ and $a^{S'_2}$ are inside the ball as shown and hence closer together than $a^{S_1}$ and $a^{S_2}$; this is a contradiction since we chose $\|a^{S'_1} - a^{S'_2}\|$ to be minimal.

This argument is formalized by choosing $n_1, n_2 \geq n$, where $n \in \mathbb{N}$ is such that

$$\frac{2 \sqrt{2}}{nA} - \frac{(A)^2 r^2}{2 \sqrt{2} (A)^2} < 0. \quad (A.1)$$

Then we show that $\|a^{S_1} - a^{S_2}\| \leq 2 \sqrt{2} / n_1 A$, where $n_1 = |S_1| = |S'_1|$. To show that the vector $a^{S_1} - a^{S_2}$ points away from $H_1$ we show that $p \cdot (a^{S_1} - a^{S_2}) \leq -Ar^2 / n_1 A$, where $p = a^{S_1} - a^{S_2}$, and symmetrically for $H_2$. We then show that $\|a^{S_1} - (a^{S_1} + a^{S_2})/2\| < \|p/2\|$, and symmetrically for $S_2$ and $S'_2$; hence $\|a^{S_1} - a^{S_2}\| < \|p\| = \|a^{S_1} - a^{S_2}\|$, which is a contradiction.

**Induction hypothesis.** Now suppose for $k = 1, 2, \ldots, m$ that for every $r > 0$ there exists $n(k, r) \in \mathbb{N}$ such that if $n_1, n_2, \ldots, n_k \geq n(k, r)$ and $S \subset \mathbb{N}$ with $|S| = \sum_{i=1}^k n_i$, then there exist $S_1, \ldots, S_k \subset S$ such that $U_i S_i = S, |S_i| = n_i$ and $\|a^{S_1} - a^S\| < r$.

Let $n(m + 1, r) = \max(n(m, r/2), n(2, r/2))$ and suppose $n_1, n_2, \ldots, n_{m+1} \geq n(m + 1, r)$ and $S \subset \mathbb{N}$ with $|S| = \sum_{i=1}^{m+1} n_i$. Then since the result is true for
k = 2, there exist S', S_{m+1} \subset S with S' \cup S_{m+1} = S, |S'| = \Sigma^m_i n_i, |S_{m+1}| = n_{m+1}
and \|a^{S'} - a^S\|, \|a^{S_{m+1}} - a^S\| < r/2. By the induction hypothesis there exist S_1, S_2, \ldots, S_m \subset S' with U^n_i S_i = S', |S_i| = n_i and \|a^{S_i} - a^{S'}\| < r/2 for i = 1, 2, \ldots, m. Thus for i \leq m,
\|a^{S_i} - a^S\| \leq \|a^{S_i} - a^{S'}\| + \|a^{S'} - a^S\| < r/2 + r/2 = r,
and therefore \|a^{S_i} - a^S\| < r for i = 1, 2, \ldots, m + 1. □

**Lemma A.5.** Let (N, V) be a game. Given φ and d_0 > 0 there exists r_0 such that for any ε > 0 if U ∈ \mathbb{R}^n and S is a coalition with

1. d(a^S, \partial Δ) \geq d_0,
2. |A^S| \geq r_0, and
3. U^S \leq (v(a^S) - φ - ε)|A^S|,
then V(A^S) > U^S + ε|A^S| so that U \notin C_φ(N, V).

**Proof.** Since V(ra)/r converges uniformly to v(a) on \{a ∈ Δ | d(a, \partial Δ) \geq d_0\}, there exists r_0 such that if d(a, \partial Δ) \geq d_0 and r \geq r_0, then V(ra) > (v(a) - φ)r. Thus conditions (1) and (2) yield

\[ V(A^S) - U^S = V(\ |A^S| a^S) - U^S > (v(a^S) - φ)\ |A^S| - (v(a^S) - φ - ε)\ |A^S| = ε\ |A^S|. \; □ \]

**Corollary.** Let (N, V) be a game and suppose that there exist A and \overline{A} such that 0 < A \leq |A^i| \leq \overline{A} for every i ∈ N. Then given φ, α and d_0 > 0, there exists n_0 such that for any ε > 0 if

1. |N| \geq n_0,
2. |S| \geq α |N|,
3. d(a^S, \partial Δ) \geq d_0, and
4. U ∈ C_α(N, V),
then u^S > v(a^S) - φ - ε, where u^S = U^S/|A^S|.

**Proof.** Let r_0 be as in Lemma A.5. Take n_0 = r_0/α A, then

\[ |A^S| \geq |S| \frac{A}{A} \geq α |N| \frac{A}{A} = α n_0 A = r_0, \]

but U ∈ C_α(N, V) so by Lemma A.5,

\[ u^S = U^S/|A^S| > v(a^S) - φ - ε. \; □ \]

**Lemma A.6.** Let (N, V) be a game. If 0 < A < |A^i| < \overline{A} for every i ∈ N, then

1. |A^S| \geq α |A^N| \Rightarrow |S| \geq α |N| \frac{A}{A}, and
2. |S| \geq α |N| \Rightarrow |A^S| \geq α |A^N| \frac{A}{A}.

**Proof:**

1. \[ α \leq \frac{|A^S|}{|A^N|} \leq \frac{|S| \overline{A}}{|N| \overline{A}} = \frac{|S|}{α} |N| \frac{A}{\overline{A}}. \]
\[
\frac{|A^S|}{|A^N|} \geq \frac{|S|}{|N|} \frac{\alpha |N|}{A} = \frac{\alpha A/\bar{A}}{\frac{\alpha}{N}} \Rightarrow |A^S| \geq \alpha |A^N| \frac{A}{\bar{A}}. \quad \square
\]

**Theorem 1.** Let \((N, V)\) be a game and suppose

1. \(V > 0\),
2. there exist \(\bar{A}\) and \(\tilde{A}\) with \(\bar{A} < \tilde{A}\) such that \(0 < A < A_i \leq \tilde{A}\) for all \(i \in N\),
3. \(a^F \in \text{int} \Delta\), and
4. \(v\) is differentiable at \(a^F\).

Let \(\alpha \in (0, 1]\) and \(\delta > 0\) be given. Then there exist \(n_0, r\) and \(\varepsilon_0 > 0\) such that if

5. \(|N| \geq n_0\),
6. \(\|a^N - a^F\| < r\),
7. \(\varepsilon \in [0, \varepsilon_0]\), and
8. \(|S| \geq \alpha |N|\),

then

\[
U \in C_{\varepsilon}(N, V), \ w \in C_{\varepsilon}(N, V) \Rightarrow |w \cdot A^S - U^S| < \delta |A^S|.
\]

**Proof.** Let \(\hat{V} : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}\) by \(\hat{V}(a) = \|A| \nu(A/|A|)\). Since \(V\) is concave, \(\hat{V}\) is also concave and since \(v\) is differentiable at \(a^F\), \(\hat{V}\) is differentiable there as well. So let \(D\hat{V}(a^F) = w^F\). We first show that there exist \(n_0, r\) and \(\varepsilon_0 > 0\) such that if \(|N| \geq n_0\), \(\|a^N - a^F\| < r\), \(\varepsilon \in [0, \varepsilon_0]\), \(|S| \geq \alpha |N|\) and \(U \in C_{\varepsilon}(N, V)\), then

\[
|w^F \cdot A^S - U^S| < \delta |A^S|.
\]

For this it suffices to show \(|w^F \cdot a^S - u^S| < \delta\), where \(u^S = U^S/|A^S|\). If \(|S| \geq \alpha |N|\), then by Lemma A.6,

\[
|A^S| \geq \gamma |A^N|, \tag{A.2}
\]

where \(\gamma = \alpha A/\bar{A}\). Note that

\[
\gamma \in (0, 1) \tag{A.3}
\]

as \(\bar{A} < \tilde{A}\) and \(\alpha \in (0, 1]\).

**Part I.** We show \(u^S = U^S/|A^S| > w^F \cdot a^S - \delta\). Suppose to the contrary that

\[
u^S \leq w^F \cdot a^S - \delta. \tag{A.4}
\]

We first show that it suffices to consider \(a^S\) outside a small neighborhood of \(a^F\):

**Claim 1.** It suffices to consider \(a^S\) such that \(\|a^S - a^F\| \geq r_0\) for some \(r_0 > 0\), and we can assume \(r_0\) is so small that

\[
\frac{1}{1 - \gamma} < \frac{2\sqrt{2}}{r_0}. \tag{A.5}
\]

The idea behind the claim can be seen in Fig. 5 in the text, where it is clear that if \(a^S\) is close to \(a^F\), then \(u^S\) is within a distance \(\delta\) of the hedonic payoff.
We therefore assume that \( ||a^s - a^F|| \geq r_0 \). We will construct a region \( D \), in the simplex, which will contain the composition of a blocking coalition. Let \( \rho^F = (a^F, v(a^F)) \). Let \( a \in \partial \Delta = \{ a \in \Delta | a_i = 0 \ \text{for some} \ i \} \) be given and let \( \rho^a = (a, w^F \cdot a - \delta) \). Let \( L(\rho^F, \rho^a, \lambda) \) be the height of the line through \( \rho^F \) and \( \rho^a \) at the point \( a^F + \lambda(a - a^F) \) (see Fig. A.2). Furthermore, let

\[
f(\rho^F, \rho^a, \lambda) = v(a^F + \lambda(a - a^F)) - L(\rho^F, \rho^a, \lambda).
\]

Since \( D \hat{v}(a^F) = w^F \) we can show that for every \( a \in \partial \Delta \) there exists \( \lambda_a \in (0, 1) \) such that \( f(\rho^F, \rho^a, \lambda) > 0 \). Since \( v \) is concave, \( f \) is concave in \( \lambda \) and continuous for \( \lambda \in [0, 1) \). Therefore for every \( a \in \partial \Delta \) there is a neighborhood of \( \lambda_a \) on which \( f(\rho^F, \rho^a, \cdot) \) is positive. We show that there is such a neighborhood, \([\lambda_0, \lambda_1]\), that is independent of \( a \). We will use the fact that \( f \) is positive on this interval to show the existence of a coalition which can \( \epsilon \)-block. From this interval we also construct the region \( D = \{ a^F + \lambda(a - a^F) \mid \lambda \in [\lambda_0, \lambda_1], a \in \partial \Delta \} \).

**Construction of the function \( f \).** We construct the linear function \( L \). Let \( a \in \Delta, \bar{a} \in \text{int} \Delta \) be distinct. We consider all points of the form \( \bar{a} + \lambda(a - \bar{a}) \) for \( \lambda \in \mathbb{R} \):

\[
[\bar{a} + \lambda(a - \bar{a})]_i = \bar{a}_i + \lambda(a_i - \bar{a}_i),
\]

so the \( r \)th coordinate of \( \bar{a} + \lambda(a - \bar{a}) \) is a linear function of \( \lambda \) which is positive for \( \lambda = 0 \). Therefore there exists a unique \( \lambda_b \geq 1 \) such that \( \bar{a} + \lambda_b(a - \bar{a}) \in \partial \Delta \). Note that \( \lambda_b = 1 \) if \( a \in \partial \Delta \). (Furthermore,

\[
0 \leq \lambda < \lambda_b \Rightarrow \bar{a} + \lambda(a - \bar{a}) \in \text{int} \Delta
\]

and

\[
\lambda > \lambda_b \Rightarrow \bar{a} + \lambda(a - \bar{a}) \notin \Delta
\]

Let \( a^b = \bar{a} + \lambda_b(a - \bar{a}) \) and let

\[
I(\bar{a}, a) = \{ \lambda \in \mathbb{R} \mid \bar{a} + \lambda(a^b - \bar{a}) \in \Delta \}.
\]

\( I(\bar{a}, a) \) is a closed interval with right endpoint 1 and 0 \( \in \text{int} I(\bar{a}, a) \) as \( \bar{a} \in \text{int} \Delta \). Let \( \rho = (a, h) \) and \( \bar{\rho} = (\bar{a}, \bar{h}) \), where \( h, \bar{h} \in \mathbb{R} \). Let \( L(\rho^F, \rho^a, \cdot) \) be

![Fig. A.2: A neighborhood where \( f(\rho^F, \rho^a, \cdot) > 0 \).](image-url)
the height of the line through $\bar{p}$ and $\rho$ above the simplex at the point $\bar{a} + \lambda(a^b - \bar{a})$, i.e.\(^{13}\)

$$L(\bar{p}, \rho, \lambda) = \bar{h} + \lambda \lambda_b (h - \bar{h}) = (1 - \lambda \lambda_b) \bar{h} + \lambda \lambda_b h. \tag{A.6}$$

Let $f(\bar{p}, \rho, \cdot): I(\bar{a}, a) \to \mathbb{R}$ by

$$f(\bar{p}, \rho, \lambda) = v(\bar{a} + \lambda(a^b - \bar{a})) - L(\bar{p}, \rho, \lambda). \tag{A.7}$$

Since $v$ is concave, $f$ is concave in $\lambda$ and continuous for $\lambda \in [0, 1)$.

Let $\rho^F = (a^F, v(a^F))$ (note that $a^F \in \text{int } \Delta$) and given $a \in \partial \Delta$, let

$$\rho^a = (a, w^F \cdot a - \delta). \tag{A.8}$$

Then $L(\rho^F, \rho^a, \lambda): I(a^F, a) \to \mathbb{R}$ by

$$L(\rho^F, \rho^a, \lambda) = v(a^F) + \lambda [w^F \cdot a - \delta - v(a^F)]$$

$$= (1 - \lambda) v(a^F) + \lambda (w^F \cdot a - \delta),$$

and $f(\rho^F, \rho^a, \lambda): I(a^F, a) \to \mathbb{R}$ by

$$f(\rho^F, \rho^a, \lambda) = v(a^F + \lambda (a - a^F)) - L(\rho^F, \rho^a, \lambda)$$

$$= v\left((1 - \lambda)a + \lambda a\right) - \left[(1 - \lambda) v(a^F) + \lambda (w^F \cdot a - \delta)\right].$$

Choice of the interval $[\lambda_0, \lambda_1]$ and construction of the region $D$.

Claim. For every $a \in \partial \Delta$ there exists $\lambda_a \in (0, 1)$ such that $f(\rho^F, \rho^a, \lambda_a) > 0$.

Given $a \in \partial \Delta$, let

$$g(a) = \sup\{\lambda \in [0, 1] | f(\rho^F, \rho^a, \lambda) > 0 \}.$$ 

Claim 2. $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in (0, g(a))$.

Since $f(\rho^F, \rho^a, \lambda_a) > 0$, $\lambda_a < g(a)$; and since $f(\rho^F, \rho^a, 0) = 0$, by concavity we get that $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in (0, \lambda_a]$. Let $\lambda_n \uparrow g(a)$ with $\lambda_n \geq \lambda_a$ and $f(\rho^F, \rho^a, \lambda_n) \geq 0$. Then by concavity, $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in [\lambda_a, \lambda_n)$ and therefore $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in [\lambda_a, g(a))$. \(\square\)

Claim 3. $\lambda^* = \inf\{g(a) | a \in \partial \Delta\} > 0$.

\(^{13}\)Since $a^b = \bar{a} + \lambda_b (a - \bar{a})$, we have $a = \bar{a} + (1/\lambda_b)(a^b - \bar{a})$. We want $L(\bar{p}, \rho, 0) = \bar{h}$ and $L(\bar{p}, \rho, 1/\lambda_b) = h$ so the slope of $L(\bar{p}, \rho, \cdot)$ must be $\lambda_b (h - \bar{h})$. 

\( f(\rho^F, \rho^a, \lambda_a) > 0 \) and \( \lambda_a \in (0, 1) \) so \( g(a) > 0 \) for all \( a \in \partial \Delta \). Suppose there exists a sequence, \( a_n \), such that \( g(a_n) \to 0 \). Let us assume without loss of generality that \( a_n \to a \in \partial \Delta \). Then for \( n \) large, \( g(a_n) < \lambda_a \), which implies that
\( f(\rho^F, \rho^{a_n}, \lambda_a) < 0 \). But then \( f(\rho^F, \rho^a, \lambda_a) \to f(\rho^F, \rho^a, \lambda_a) \), which is a contradiction because \( f \) is continuous in \( \rho^a \) (since \( \lambda_a < 1 \) and \( v \) is continuous on \( \text{int} \Delta \)).

We now define the region \( D = \{ a^F + \lambda(a - a^F) \mid \lambda \in [\lambda_0, \lambda_1], a \in \partial \Delta \} \) by choosing the values \( \lambda_0 \) and \( \lambda_1 \). We also choose \( \gamma_0 \) and \( \gamma_1 \) such that if \( \gamma_0 \leq \| a^D - a^F \| \leq \gamma_1 \), then \( a^D \in D \).

Since \( a^F \in \text{int} \Delta \), we can choose \( r_i > 0 \) such that \( r_i < \inf \{ \| a - a^F \| : a \in \partial \Delta \} \).

Let
\[
\alpha_0 = \sup \{ \| a - a' \| : a \in \partial \Delta, \| a' - a^F \| \leq r_i \}, \tag{A.9}
\]
\[
\alpha_1 = \inf \{ \| a - a' \| : a \in \partial \Delta, \| a' - a^F \| \leq r_i \}. \tag{A.10}
\]

Note that
\[
\| a' - a^F \| \leq r_i \Rightarrow a' \in \partial \Delta, \tag{A.11}
\]
so that \( \alpha_1 > 0 \). We choose \( \lambda_1 \in (0, \lambda^*) \) and let \( \gamma_1 = \lambda_1 \alpha_1 \). Let \( f: [0, 1] \times [1/(1 - \gamma), 2\sqrt{2}/r_0] \to \mathbb{R} \) by
\[
f(x, y) = (1 - x)/(y - x). \tag{A.12}
\]
(Recall \( \gamma \in (0, 1) \) by (A.3) and \( 1/(1 - \gamma) < 2\sqrt{2}/r_0 \) by (A.5).) \( f \) is continuous and therefore uniformly continuous so there exists \( \psi > 0 \) such that
\[
x \leq \psi \Rightarrow |f(x, y) - f(0, y)| \leq \gamma_1/2\sqrt{2} \text{ for any } y. \tag{A.13}
\]

Since
\[
\frac{\partial f}{\partial x} = \frac{(y - x)(-1) - (1 - x)(-1)}{(y - x)^2} = \frac{x - y + 1 - x}{(y - x)^2} = \frac{1 - y}{(y - x)^2} < 0,
\]
f is decreasing in \( x \). We choose \( k \in \mathbb{N} \) such that
\[
2 \tilde{A}/kA \leq \min \{ \psi, 1 \} \tag{A.14}
\]
and let
\[
L = \min_y \left[ f(0, y) - f\left( \frac{A}{2kA}, y \right) \right] > 0. \tag{A.15}
\]
We choose \( \lambda_0 \in (0, \lambda_1) \) such that
\[
\gamma_0 \leq Lr_0/8 \tag{A.16}
\]
and let $\gamma_0 = \lambda_0 \alpha_0$.

**Claim 4.** There exists $M > 0$ such that $f(\rho^F, \rho^a, \lambda) \geq M$ for $\lambda \in [\lambda_0, \lambda_1]$, $a \in \partial \Delta$.

As a function of $a$ and $\lambda$, $f$ is continuous on $\partial \Delta \times [\lambda_0, \lambda_1]$ since $\nu$ is continuous on $\text{int} \Delta$ and $\lambda_1 < \lambda^* < 1$. Thus it suffices to show $f(\rho^F, \rho^a, \lambda) > 0$ for $a \in \partial \Delta$, $\lambda \in [\lambda_0, \lambda_1]$. And since $f$ is concave in $\lambda$, it suffices to show $f(\rho^F, \rho^a, \lambda_i) > 0$ for $i = 0, 1$ and $a \in \partial \Delta$. Suppose there exist $a_n \in \partial \Delta$ such that $f(\rho^F, \rho^a, \lambda_i) \to 0$. Let us assume without loss of generality that $a_n \to a$. Then $f(\rho^F, \rho^a, \lambda_i) = 0$. But this contradicts Claims 2 and 3. □

We next use Lemma A.4 to partition $N \setminus S$ into subcoalitions $S_1, \ldots, S_k$ such that $\gamma_0 \leq \| a^N \setminus S_i - a^F \| \leq \gamma_1$. We have chosen $\gamma_0$ and $\gamma_1$ such that $f(\rho^F, \rho^a, \cdot) \geq M > 0$ at points corresponding to this region of the simplex. It will therefore follow that one of the coalitions $N \setminus S_j$ can block.

If $k$ is large, then $a^N \setminus S_i \sim a^N$, and if $|N|$ is large, then $a^N \sim a^F$, so $a^N \setminus S_i \sim a^F$. Thus we can obtain the inequality involving $\gamma_1$ by choosing $k$ and $|N|$ sufficiently large. But no matter how large $k$ is, removing $S_j$ from $N$ moves the mean, $a^N$, at least some distance away from $a^N \sim a^F$ to $a^N \setminus S_j$. This is because by Lemma A.4, $a^S \sim a^N \setminus S$ and $a^S \setminus S$, $a^S$ are bounded away from $a^N \sim a^F$ because $S$ is a significant portion of the population ($|S| > \alpha |N|$). Therefore $a^N \setminus S_i \neq a^F$ and we obtain the inequality involving $\gamma_0$ by having chosen $\gamma_0$ sufficiently small. These assertions are formalized in the following claims, ending with Claim 7:

**Claim.** $K = \{ a^I + \lambda(a - a^I); \| a^I - a^F \| \leq r_1, \lambda \in [\lambda_0, \lambda_1], a \in \partial \Delta \}$ is compact.

Since $\hat{V}$ is concave, it is Lipschitz (Roberts and Varberg, 1973, p. 93) on the compact set $K$, which is in the interior of $\mathbb{R}^r_+ \setminus \{0\}$ by (A.11). Let $C > 0$ be a Lipschitz constant for $\hat{V}$ on $K$. Then since $\hat{V} = \nu$ on $\Delta$,

$$ |\nu(a) - \nu(a')| \leq C \| a - a' \| \quad \text{for } a, a' \in K. \quad \text{(A.17)} $$

We choose $r > 0$ such that

$$ r \leq \min \left\{ \frac{\gamma_1}{8}, \frac{Lr_0}{16}, \frac{r_0}{2}, \frac{M}{8C}, \frac{r_1}{2} \right\}. \quad \text{(A.18)} $$

We assume further that $r$ is so small that

$$ 2r \| w^F \| \leq \delta/2 \quad \text{(A.19)} $$

and, since $\nu$ is continuous at $a^F$, that

$$ \| a^N - a^F \| \leq r \Rightarrow |\nu(a^N) - \nu(a^F)| \leq \min \left\{ \frac{\delta}{4}, \frac{M}{8} \right\}. \quad \text{(A.20)} $$
We assume
\[
\| a^N - a^f \| < r. \tag{A.21}
\]

Claim 5. \( a^s \) is bounded away from \( a^N \): \( \| a^s - a^N \| \geq r_0 - r \).

Claim. \( | A^{N \setminus S} | \) is neither trivially small nor very large: \( (r_0 - r) / \sqrt{2} \leq \frac{1}{| A^{N \setminus S} |} < 1 - \gamma \).

Let \( | N \setminus S | = kq + l \), where \( q, l \in \mathbb{Z} \) and \( 0 \leq l < k \). By Lemma 6, \( | N \setminus S | \geq (r_0 - r) / \sqrt{2} | N | A/\widetilde{A} \), so that if \( | N | \) is large, then \( q \) is large. We choose \( n_2 \) such that if \( | N | \geq n_2 \), then Lemma 5 applies with \( k \) and \( r \) as above and \( n_1 = q \) or \( q + 1 \). Let \( S_1, \ldots, S_k \) be the partition of \( N \setminus S \) from Lemma 5. Then
\[
\| a^{S_i} - a^{N \setminus S} \| < r. \tag{A.22}
\]

We assume without loss of generality that \( u^{S_i} \geq u^{S_i} \) for \( i = 1, \ldots, k \).

Claim 6. \( a^{N \setminus S_i} \) is not too far away from \( a^f \): \( \| a^{N \setminus S_i} - a^f \| < \gamma_1 - r \).

Claim 7. \( a^{N \setminus S_1} \) is not too close to \( a^f \): \( \| a^{N \setminus S_1} - a^f \| > L r_0 / 4 > \gamma_0 \).

To complete the proof we show that \( N \setminus S_i \) can \( \epsilon \)-block. By Claim 4 we know that \( f(\rho^f, \rho^a, \lambda) \geq M \) for \( \lambda \in [\lambda_0, \lambda_1] \), \( \alpha \in \partial \Delta \). If \( a^{N \setminus S_i} = a^f + \lambda(a - a^f) \) for some \( \lambda \in [\lambda_0, \lambda_1] \), \( \alpha \in \partial \Delta \), and if \( \phi \) and \( \epsilon \) are small, then \( f(\rho^f, \rho^a, \lambda) - \phi - \epsilon = v(a^{N \setminus S_i}) - \phi - \epsilon = v(a^f + \lambda(a - a^f)) - \phi - \epsilon \geq L(\rho^f, \rho^a, \lambda) \). To show that \( N \setminus S_i \) can block, by the Corollary to Lemma 5, it suffices to show that \( u^{N \setminus S_i} \leq L(\rho^f, \rho^a, \lambda) \). In fact, it suffices to show that \( u^{N \setminus S_i} \) lies below a line that is close to \( L(\rho^f, \rho^a, \lambda) \), namely \( L(\rho^l, \rho^b, \lambda) \), where \( \rho^l \) is close to \( \rho^f \) and \( \rho^b \) is a boundary point chosen so that for an appropriate \( \lambda_0 \) the line is defined at the point \( a^{N \setminus S_i} \).

Let \( p^s = (a^s, u^s) \), \( p^s = (a^{N \setminus S}, u^{N \setminus S}) \) and \( p^N = (a^N, u^N) \). By Lemma 1,
\[
p^N = (1 - \beta) p^s + \beta p^{N \setminus S}, \quad \beta = \| A^{N \setminus S} \| / | A^N | .
\]
Let
\[
\bar{\rho} = (\bar{a}, \bar{u}) = (a^{N \setminus (S \cup S_i)}, u^{N \setminus S}) \tag{A.23}
\]
and let
\[
\rho^l = (\alpha^l, u^l) = (1 - \beta) p^s + \beta \bar{\rho}. \tag{A.24}
\]
By Lemma 4, \( \rho^l \sim p^N \setminus S \) and so \( \rho^l \sim p^N \) as both are the same convex combination of points that are close. We show that \( p^N \sim p^f \) and therefore \( \rho^l \sim \rho^f \). Then by continuity, \( f(\rho^l, \rho^a, \lambda) \geq M / 2 \) for \( \lambda \in [\lambda_0, \lambda_1] \) and \( \alpha \in \partial \Delta \). Hence, if \( \phi \) and \( \epsilon \) are small, then \( f(\rho^l, \rho^a, \lambda) - \phi - \epsilon \geq 0 \) for \( \lambda \in [\lambda_0, \lambda_1] \), \( \alpha \in \Delta \). This will yield a contradiction to the corollary to Lemma 5.

In the following set of claims, ending in Claim 9, we show that \( f(\rho^l, \rho^a, \lambda) \geq M / 2 \) for \( \lambda \in [\lambda_0, \lambda_1] \), \( \alpha \in \partial \Delta \). We then complete the proof by defining the line.
\( L(\rho^l, \rho^{b'}, \lambda) \) and showing that since \( u^{N\setminus S_i} \leq L(\rho^l, \rho^{b'}, \lambda) \), the coalition \( N\setminus S_i \) can block.

**Claim 8.** \( \| a' - a^F \| < 2r. \)

Let \( \phi = \epsilon_0 = \min(\delta/8, M/16) \) and assume \( \epsilon \in [0, \epsilon_0] \). Then

\[
\phi + \epsilon \leq \min\left( \frac{\delta}{4}, \frac{M}{8} \right).
\]

Let \( d_0 = d(a^F, \partial\Delta) - (\gamma_1 - 2r) \). Then

**Claim.** \( d(a^{N\setminus S_i}, \partial\Delta) \geq d_0 > 0. \)

Thus by the corollary to Lemma A.5, there exists \( n_3 \) such that

\[
|N| \geq n_3, \quad |S| \geq \alpha \cdot |N|, \quad d(a^S, \partial\Delta) \geq d_0,
\]

\[
U \in C_{a}(N, V) \Rightarrow u^S > v(a^S) - \phi - \epsilon.
\]  

We assume \( |N| \geq \max(n_1, n_2, n_3) \).

**Claim 9.** \( |v(a^F) - u^N| < \min(\delta/2, M/4). \)

**Claim 10.** \( f(\rho^l, \rho^a, \lambda) \geq M/2 \) for \( \lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta. \)

By Claim 8 and (A.18), \( \| a' - a^F \| < 2r \leq r_1 \) so that we can apply (A.17) to get

\[
|f(\rho^l, \rho^a, \lambda) - f(\rho^F, \rho^a, \lambda)|
\]

\[
= |v(a' + \lambda(a - a')) - L(\rho^l, \rho^a, \lambda) - v(a^F + \lambda(a - a^F))|
\]

\[
+ L(\rho^F, \rho^a, \lambda) \text{ by (7)}
\]

\[
\leq |v((1 - \lambda)a' + \lambda a) - v((1 - \lambda)a^F + \lambda a)|
\]

\[
+ |L(\rho^F, \rho^a, \lambda) - L(\rho^l, \rho^a, \lambda)|
\]

\[
\leq C(1 - \lambda) \| a' - a^F \| + |(1 - \lambda)v(a^F) + \lambda(w^F \cdot a - \delta)
\]

\[
- [(1 - \lambda)u' + \lambda(w^F \cdot a - \delta)] \text{ by (A.17), (A.6) and (A.8)}
\]

\[
\leq C \| a' - a^F \| + (1 - \lambda) |v(a^F) - u^F|
\]

\[
= C \| a' - a^F \| + (1 - \lambda) |v(a^F) - u^N| \text{ by (A.24)}
\]

\[
< C2r + \frac{M}{4} \leq \frac{M}{4} + \frac{M}{4} = \frac{M}{2}
\]

by Claim 8, Claim 9 and (A.18). So by Claim 4, \( f(\rho^l, \rho^a, \lambda) \geq M/2 \) for \( \lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta. \) \( \square \)
Therefore $f(\rho', \rho^b, \lambda) - \phi - \epsilon \geq M/2 - M/8 > 0$ for $\lambda \in [\lambda_0, \lambda_1]$, $a \in \partial \Delta$ by (A.25).

In the following claims we show that $N \setminus S_i$ can $\epsilon$-block, since $u_{N \setminus S_i} \leq L(\rho', \rho^b', \lambda)$, and this will complete the proof of Part I. Let $p^{N \setminus (S \cup S_i)} = (a^{N \setminus (S \cup S_i)}, u^{N \setminus (S \cup S_i)})$, $p^{N \setminus S_i} = (a^{N \setminus S_i}, u^{N \setminus S_i})$ and $\beta' = |A^{N \setminus (S \cup S_i)}|/|A^{N \setminus S_i}|$. We have from Lemma 1 that

$$p^{N \setminus S_i} = (1 - \beta') p^S + \beta' p^{N \setminus (S \cup S_i)}. \tag{A.27}$$

Let

$$\rho^D = (a^D, u^D) = (1 - \beta') p^S + \beta' \tilde{\rho}, \tag{A.28}$$

where $\tilde{\rho}$ is given by (A.23). We have $a^D = a^{N \setminus S_i}$, and it can be shown that $\beta > \beta'$. In Claim 11 we extend the line segment from $a^{N \setminus (S \cup S_i)}$ to $a^S$ until it meets $\partial \Delta$ at a point $b'$ (see Fig. A.3) and choose $\lambda_D$ such that $a^{N \setminus S_i} = a^D = (1 - \lambda_D) a^I + \lambda_D b'$, $\lambda_D \in [\lambda_0, \lambda_1]$. Thus, by the remark after Claim 10, $f(\rho', \rho^b, \lambda_D)$ $- \phi - \epsilon \geq 0$. By the application of Lemma A.4, $N \setminus S = \bigcup_{i=1}^k S_i$. We can assume that $u^{S_i} \geq u^S_i$ for $i = 1, 2, \ldots, k$. Therefore $u^{N \setminus (S \cup S_i)} \leq u^{N \setminus S_i}$ and so $p^{N \setminus (S \cup S_i)}$ lies below $\tilde{\rho}$ and hence $p^{N \setminus S_i}$ lies below $\rho^D$. We then show that $L(\rho', \rho^b, \cdot)$ lies above $p^S$ and hence above $\rho^D$, i.e. $L(\rho', \rho^b, \lambda_D) \geq u^D \geq u^{N \setminus S_i}$. But then

$$0 \leq f(\rho', \rho^b, \lambda_D) - \phi - \epsilon $$

$$= v(a^D) - L(\rho', \rho^b, \lambda_D) - \phi - \epsilon $$

$$\leq v(a^{N \setminus S_i}) - u^{N \setminus S_i} - \phi - \epsilon. $$

This contradicts the corollary to Lemma A.5 (which applies as $N \setminus S_i \supset S$) and thus completes the proof.

**Claim.** $\gamma_0 \leq ||a^D - a^I|| \leq \gamma_1$.

**Claim 11.** There exist $b' \in \partial \Delta$ and $\lambda_D \in [\lambda_0, \lambda_1]$ such that $a^D = (1 - \lambda_D) a^I + \lambda_D b'$ and $f(\rho', \rho^b, \lambda_D) - \phi - \epsilon > 0.$
By (A.24) we know that
\[ a' = (1 - \beta') a^s + \beta \bar{a}. \] (29)
And by (A.28) we know that
\[ a^D = (1 - \beta') a^s + \beta' \bar{a}. \] (30)
Subtracting (A.29) from (A.30) we get
\[ a^D - a' = (\beta' - \beta)(\bar{a} - a^s), \]
and substituting into (A.29) we get
\[ a^s = a' + \beta(a^s - \bar{a}) = a' + \frac{\beta}{\beta - \beta'}(a^D - a'). \] (31)
Since \( \beta > \beta' \), and by the definition of \( \lambda_D \) in the proof of Claim 11, \( \beta/(\beta - \beta') \)
\( \leq 1/\lambda_D \) which implies that \( \lambda_D \beta/(\beta - \beta') \leq 1 \). Let \( \lambda_s = \lambda_D \beta/(\beta - \beta') \).

Claim 12. \( a^s = (1 - \lambda_s) a' + \lambda_s b' \).

Claim. \( u^N = u' > w^F \cdot a' - \delta \).

Claim 13. \( L(\rho', \rho^{b'}, \lambda_s) > u^N \).

Claim. \( p^D = (1 - \lambda_D) \rho' + \lambda_D p^s \).

Therefore
\[ u^D = (1 - \lambda_D) u' + \lambda_D u^s = (1 - \lambda_D) u^N + \lambda_D u^s \]
because \( u' = u^N \). Furthermore,
\[ a^D = (1 - \lambda_D) a' + \lambda_D a^s \]
\[ = (1 - \lambda_D) a' + \lambda_D [(1 - \lambda_s) a' + \lambda_s b'] \text{ by Claim 12} \]
\[ = (1 - \lambda_D \lambda_s) a' + \lambda_D \lambda_s b'. \]
Therefore \( \lambda_D \lambda_s = \lambda_D \) by Claim 11 (\( a' \neq b' \) since \( a' \notin \delta \Delta \) by the proof of Claim 11):
\[ u^D = (1 - \lambda_D) u^N + \lambda_D u^s = (1 - \lambda_D) u^N + \lambda_D L(\rho', \rho^{b'}, \lambda_s) \text{ by Claim 13} \]
\[ = (1 - \lambda_D) u^N + \lambda_D [(1 - \lambda_s) u^N + \lambda_s (w^F \cdot b' - \delta)] \]
by (A.6) and (A.8)
\[ = (1 - \lambda_D \lambda_s) u^N + \lambda_D \lambda_s (w^F \cdot b' - \delta) \]
\[ = L(\rho', \rho^{b'}, \lambda_D \lambda_s) = L(\rho', \rho^{b'}, \lambda_D), \]
i.e. \( L(\rho^I, \rho^B, \cdot) \) is above \( u^D \) at \( a^D \) (see Fig. A.3). We have chosen \( S_i \) so that 
\[ u^{N\setminus S_i} = (1 - \beta') u^S + \beta' u^{N\setminus(S \cup S_i)} \]
by (A.27)
\[ \leqslant (1 - \beta') u^S + \beta' u^{N\setminus S} = u^D \]
by (A.28) and (A.23)
\[ < L(\rho^I, \rho^B, \lambda_D). \]
Furthermore, by Claim 11,
\[ 0 < f(\rho^I, \rho^B, \lambda_D) - \phi - \epsilon = v((1 - \lambda_D) a^I + \lambda_D b^I) \]
\[ - L(\rho^I, \rho^B, \lambda_D) - \phi - \epsilon < v(a^D = a^{N\setminus S_i}) - u^{N\setminus S_i} - \phi - \epsilon. \]
Thus, \( u^{N\setminus S_i} < v(a^{N\setminus S_i}) - \phi - \epsilon. \) This contradicts (A.26), which applies because \( S_i \subset N \setminus S \Rightarrow N \setminus S_i \supset S \Rightarrow |N \setminus S_i| \geq |S| \geq \alpha |N|. \)
We have thus shown that there exist \( r, \epsilon_0 > 0 \) such that if \( |N| \geq \max\{n_1, n_2, n_3\}, \|a^N - a^0\| \leq r, \epsilon \in [0, \epsilon_0], |S| \geq \alpha |N| \) and \( U \in C_\epsilon(N, V) \), then \( w^F \cdot A^S - U^S < \delta \|
A^S\| \).

**Part II.** We show that \( u^S = U^S / |A^S| < w^F \cdot a^S + \delta. \)

Suppose to the contrary that 
\[ u^S \geq w^F \cdot a^S + \delta. \]
Let \( \beta = |A^{N\setminus S}| / |A^N| \). Then by (A.2) we get \( 1 - \beta \geq \gamma \), which implies that \( \beta \leq 1 - \gamma \).

Let \( f(x) = (1 - x)/x. \) Then \( f'(x) = [-x - (1 - x)]/x^2 = -1/x^2 < 0. \) Therefore \( f \) is decreasing and so (assuming \( \beta \neq 0 \))
\[ \frac{1 - \beta}{\beta} \geq \frac{1 - (1 - \gamma)}{1 - \gamma} = \frac{\gamma}{1 - \gamma} \Rightarrow \frac{1 - \beta}{\beta} \leq \frac{\gamma}{1 - \gamma}. \]

Since \( D\hat{V}(a^F) = w^F, w^F \) is a supported hyperplane for \( \hat{V} \) at \( a^F \), i.e. \( w^F \cdot a^F = \hat{V}(a^F) \) and \( w^F \cdot A \geq \hat{V}(A) \) for every \( A \in \mathbb{R}_+^T \setminus \{0\} \) (Roberts and Varberg, 1973, p. 115). On the simplex this gives \( w^F \cdot a^F = v(a^F) \) and \( w^F \cdot a \geq v(a) \) for every \( a \in A \), and so by Lemma 1,
\[ w^F \cdot a^N \geq v(a^N) \geq u^N = (1 - \beta) u^S + \beta u^{N\setminus S} \]
\[ \geq (1 - \beta)(w^F \cdot a^S + \delta) + \beta u^{N\setminus S}. \]
This implies
\[ \beta u^{N\setminus S} \leq w^F \cdot a^N - (1 - \beta) w^F \cdot a^S - (1 - \beta) \delta \]
\[ = w^F \cdot [(1 - \beta) a^S + \beta a^{N\setminus S}] - (1 - \beta) w^F \cdot a^S - (1 - \beta) \delta \]
\[ = \beta w^F \cdot a^{N\setminus S} - (1 - \beta) \delta. \]
Hence

\[ u^N \preceq w^N \cdot a^N \preceq \frac{1 - \beta}{\beta} \delta \leq w^N \cdot a^N \preceq \frac{\gamma}{1 - \gamma} \delta. \tag{A.33} \]

(Note that \( \beta \neq 0 \), since otherwise \( S = N \), which implies that \( u^S = u^N \leq w^N \cdot a^N = w^F \cdot a^S \), and this contradicts (A.32).) Now if there is a positive lower bound for \( \beta \), then by Lemma A.6, \( |N \setminus S| \) would constitute a non-trivial fraction of the population and so the argument in Part I, with \( \gamma \delta/(1 - \gamma) \) in place of \( \delta \), will apply to \( N \setminus S \) to produce a contradiction. Thus we will show that \( \beta \) has a positive lower bound. Recall that \( p^N = (a^N, u^N) \) and \( p^{N \setminus S} = (a^{N \setminus S}, u^{N \setminus S}) \).

Claim. \( \| p^N - p^{N \setminus S} \| \leq \sqrt{2 + (\| w^N \| + \varepsilon)^2} \).

Claim. \( \| p^S - p^N \| \geq \delta/\| (w^F, -1) \| \).

Claim. \( \beta = \| p^N - p^S \| / \| p^{N \setminus S} - p^S \| \).

Thus,

\[ \beta = \frac{\| p^N - p^S \| / \| p^{N \setminus S} - p^S \|}{\| (w^F, -1) \|} \geq \sqrt{2 + (\| w^N \| + \varepsilon)^2}, \]

and so \( \beta \) has a positive lower bound.

This shows that there exist \( n_0, r, \varepsilon_0 > 0 \) such that if \( |N| \geq n_0, \| a^N - a^F \| \leq r, \varepsilon \in [0, \varepsilon_0], |S| \geq \alpha |N| \) and \( U \in C_e(N, V) \), then

\[ |w^F \cdot a^S - U^S| < \frac{\delta}{2} |A^S|. \]

If \( w \in C_e^H(N, V) \), then \( (w \cdot A^1, \ldots, w \cdot A^n) \in C_e(N, V) \) so that

\[ |w \cdot A^S - U^S| \leq |w \cdot A^S - w^F \cdot A^S| + |w^F \cdot A^S - U^S| < \frac{\delta}{2} |A^S| + \frac{\delta}{2} |A^S| = \delta |A^S|. \square \]

References


Scotchmer, S., 1994, Externality pricing in club economies, Ricerche Economiche, in press.


